THE APPLICATIONS OF MATHEMATICS IN METEOROLOGY*

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1. Introduction. The aim of this paper is to enumerate the mathematical prerequisites which are needed to take up the study of modern meteorology. The discussion will be confined to the more elementary topics. This limitation appears especially justified at the present time because the five institutions which offer complete training courses in meteorology have shortened and simplified their curricula, owing to the large and urgent demand for meteorologists in the armed services. At the Massachusetts Institute of Technology, for instance, meteorology was taught in a two-year course to graduate students. After a successful completion of this course, the students received the degree of Master of Science. At present the course is reduced to a period of nine months at all the institutions which offer these meteorological courses. The student body is made up mainly of Aviation Cadets, and further of Naval Ensigns and some civilian students. The Aviation Cadets are, after successful completion of the course, commissioned as Second Lieutenants in the Army Air Corps. At present, only a small fraction of the civilian students remain for two years in order to qualify for the Master's Degree.

The most important practical function of the meteorologist in wartime, and, for that matter, in peacetime, is weather forecasting. Consequently, the meteorological courses, especially now, stress those problems which are closely related to the analysis of the present weather and to the prognosis of the future weather. Since the advances of the forecasting technique are largely due to the introduction of physical principles, in particular due to the application of the appropriate laws of thermodynamics and hydrodynamics, a knowledge of these subjects and the mathematical tools necessary for their understanding is indispensable for the modern meteorologist. The branch of meteorology which studies the application of the laws of thermodynamics and hydrodynamics to meteorological problems is called "dynamic meteorology," and it is mainly dynamic meteorology where a knowledge of some mathematics is necessary.

It may be mentioned here that statistical methods are also widely used in meteorology. The statistical approach permits one to overcome to a certain extent the handicap which arises out of the fact that it is impossible to conduct controlled experiments in meteorology. But the use of mathematical statistics in meteorology will not be discussed here since statistical methods are not required in daily forecasting, even though the results of statistical investigations may be of great assistance to the forecaster.

2. The hydrodynamic equations in meteorology. A study of dynamic meteorology would logically begin with the hydrodynamic equations. In the introductory courses in dynamic meteorology it is, however, preferable to begin in a

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more elementary fashion by considering simple cases of balanced motions. But for the sake of completeness we shall here start out with the hydrodynamic equations.

Let $\mathbf{v}$ denote the velocity vector of a particle of air, $\boldsymbol{\Omega}$ the vector of the earth's rotation, ρ the density, p the pressure, g the acceleration of gravity, $\boldsymbol{\kappa}$ the unit vector directed vertically upward. Then the equations of motion and of continuity are:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + 2\boldsymbol{\Omega} \times \mathbf{v} = -\frac{1}{\rho}\nabla p - \boldsymbol{\kappa} g, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0. \tag{2}$$

Counting the vector equation (1) as three equations, *four* equations are available to determine the five unknown variables, *viz.*, the three velocity components u, v, w, the pressure p, and the density ρ. The number of equations is equal to the number of unknown variables only if the fluid is incompressible because the density ρ can, under this condition, be regarded as a known parameter. But in a study of atmospheric motions the compressibility has in many cases to be taken into account. A fifth equation is then required to complete the system of equations. Such an equation can be obtained from physical considerations. Often it may be assumed that the relation between pressure and density changes follows the adiabatic law of compression and expansion, so that the adiabatic relation would complete the system of equations. After the introduction of appropriate boundary conditions for a particular problem, the motion is fully determined by the equations, the boundary conditions and the initial state of the fluid. The assumption of adiabatic motion has, for instance, been made mostly in the study of atmospheric wave motions.

In elementary instruction in dynamic meteorology such advanced problems are not taken up. The topics which are discussed comprise rather simple matters which are nevertheless of great practical importance. Some of these topics will here be reviewed briefly in order to show the approach of dynamic meteorology to its problems. A more detailed discussion can be found in the text books on dynamic meteorology.*

3. The geostrophic wind. One of the variables which is entered on the weather maps is the atmospheric pressure reduced to sea level. Its horizontal distribution is represented by the "isobars," that is, by lines connecting points of equal pressure. A study of the weather maps shows that the wind velocity is approximately proportional to the horizontal pressure gradient and that, in the northern hemisphere, the wind direction is such that, facing downwind, one has the lower pressure to the left, the higher pressure to the right. These empirical results can easily be explained with the aid of the equation of motion (1). The

* D. Brunt, Physical and Dynamical Meteorology, Second Edition, Cambridge, 1939.
B. Haurwitz, Dynamic Meteorology, New York, 1941.

vertical component of the motion may be neglected, and it may be assumed that the horizontal wind components u and v are independent of space and time. Then, from (1)

$$(3) \qquad -2\Omega_z v = -\frac{1}{\partial}\frac{\partial p}{\partial x}, \qquad 2\Omega_z u = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

where $\Omega_z = \omega \sin \phi$, the vertical component of the vector of the earth's rotation at the place under consideration, ω is the angular velocity of the earth's rotation, and ϕ is the latitude. If, for the sake of convenience, the coordinate system is rotated around the z-axis until the y-axis coincides with the wind direction, it follows that

$$(4) \qquad v = \frac{1}{\rho}\frac{1}{2\omega \sin \phi}\frac{\partial p}{\partial x}, \qquad u = 0.$$

Hence the velocity of the wind is proportional to the pressure gradient, and the wind direction is perpendicular to the pressure gradient. If the pressure increases in the positive x-direction, v is positive, that is, the wind blows in the direction of the positive y-axis. It follows that an observer facing downwind (in the direction of the positive y-axis) has the higher pressure to his right (in the direction of the positive x-axis), the lower pressure to his left. This result holds for the northern hemisphere. In the southern hemisphere ϕ is negative, since the latitude has to be counted negative. Consequently, the low pressure is to the left of the wind in the southern hemisphere.

In the layers next to the ground, up to about 3000 feet, the geostrophic wind relation is not as well satisfied as at higher levels, owing to the effect of the friction at the earth's surface. The wind here does not blow perpendicular to the pressure gradient but makes a smaller angle with it.

With the aid of the geostrophic wind relation one can frequently plot the pressure distribution more accurately than would otherwise be possible, especially over the oceans where, even in peacetime, observations are fewer and more widely apart than over land. The influence of friction is much smaller over the water than over land and much more uniform, so that a correction for friction can easily be applied to the geostrophic wind relation. The intensity and direction of the wind indicate according to the geostrophic relation the direction of the isobars and their distance in the environment of the point of observation. If pressure and wind are given at places which are far apart, a more accurate representation of the pressure distribution may thus be obtained by noting not only the pressure at each station, but also the wind velocity which indicates the distance between the isobars in the neighborhood of each station, this distance being equivalent to the pressure gradient.

The geostrophic relation permits one, furthermore, to obtain a fairly reliable estimate of the wind velocity above the layer of frictional influence, that is, at levels of 3000 feet, approximately. Such estimates are particularly desirable when direct observations of the winds at higher levels are impossible, due to overcast skies. At greater heights wind estimates based on the distribution of the surface pressure become, however, less satisfactory, since the pressure gradient changes as a rule with the altitude. But it is possible to take the effect of this vertical variation of the pressure gradient on the wind into account by means of the so-called thermal wind.

4. The thermal wind. To determine this variation of the geostrophic wind with the height the equations (3) have to be differentiated with respect to the height coordinate z. Since not the density but the pressure and temperature of the air are measured in meteorology, it is appropriate to eliminate the density from the two equations (3) with the aid of the equation of state for ideal gases:

$$\rho = \frac{p}{RT}$$

where R is the gas constant for air, T the absolute temperature. The use of this equation is justified, since the air may be treated as an ideal gas at the temperatures and pressures which occur in the atmosphere. Hence

$$\begin{aligned} \frac{\partial}{\partial z}\left(\frac{v}{T}\right) &= \frac{R}{2\Omega_z}\,\frac{\partial}{\partial z}\left(\frac{\partial\, ln\, p}{\partial x}\right), \\ \frac{\partial}{\partial z}\left(\frac{u}{T}\right) &= -\frac{R}{2\Omega_z}\,\frac{\partial}{\partial z}\left(\frac{\partial\, ln\, p}{\partial y}\right). \end{aligned} \tag{4}$$

It is convenient to write the equations in this form, since T is in general a function of z. Since hydrostatic equilibrium can be assumed in vertical direction

$$g = -\frac{1}{\rho}\,\frac{\partial p}{\partial z},$$

or substituting from the equation of state

$$\frac{g}{T} = -R\,\frac{\partial\, ln\, p}{\partial z}. \tag{5}$$

On the right side of the two equations (4) the order of the partial differentiations may be interchanged. It follows with the aid of (5) that

$$\begin{aligned} \frac{\partial}{\partial z}\left(\frac{v}{T}\right) &= \frac{g}{2\Omega_z}\,\frac{1}{T^2}\,\frac{\partial T}{\partial x}, \\ \frac{\partial}{\partial z}\left(\frac{u}{T}\right) &= -\frac{g}{2\Omega_z}\,\frac{1}{T^2}\,\frac{\partial T}{\partial y}. \end{aligned} \tag{6}$$

These two relations show that the variation of the geostrophic wind in vertical direction depends on the horizontal temperature gradient, a quantity which can easily be obtained from the observations. In the lower part of the atmosphere, the troposphere, the temperature decreases at a linear rate with the altitude

$$T = T_0 - mz.$$

If it is assumed that the rate m of the vertical variation of the temperature is independent of the horizontal coordinates, equations (6) can be integrated. Integrating between the surface, indicated by the subscript "0," and the level z

$$(7) \qquad \begin{aligned} v &= v_0 \frac{T}{T_0} + \frac{gz}{2\Omega_z} \frac{1}{T_0} \frac{\partial T_0}{\partial x}, \\ u &= u_0 \frac{T}{T_0} - \frac{gz}{2\Omega_z} \frac{1}{T_0} \frac{\partial T_0}{\partial y}. \end{aligned}$$

These relations give a fairly reliable estimate of the upper winds from observations of the surface wind and of the temperature field if direct observations of the upper winds are not available, for instance, due to low clouds. Under such weather conditions, reliable estimates of the upper winds are particularly important for air navigation.

If, on the other hand, the vertical wind distribution is known but the temperature distribution unknown, equations (7) can be used to obtain some information about the horizontal temperature distribution. As an example, suppose that the wind is from the south at the surface and from the west at 10,000 feet, while its intensity is the same at both heights. Choosing the x-axis positive eastwards, the y-axis positive northwards, the assumptions imply that

$$\begin{aligned} u_0 &= 0, \qquad & v_0 &> 0, \\ u &> 0, \qquad & v &= 0, \end{aligned}$$

and hence $\partial T_0/\partial y < 0$, $\partial T_0/\partial x < 0$. The numerical values of the two components of the horizontal temperature gradient are almost equal under the above conditions, since the ratio T/T_0 is not much different from unity. It follows that the temperature gradient is directed to the southwest. Since the wind turns from a southerly direction at the ground to a westerly direction aloft, it can be expected that the temperature at the place of observation will rise, due to the advection of warmer air.

Such forecasts based on observations made at a single station are of special importance in wartime when a meteorologist, on a warship for instance, may not be able to receive weather reports from other localities. He will then have to base his forecasts entirely on observations made at his own post. This procedure is called "spot forecasting." Under such circumstances when as much information as possible has to be obtained from very few observations, conclusions based on dynamical reasoning are of special importance.

5. The variation of the wind in the frictional layer. It has been mentioned previously that in the layer from the surface up to a few thousand feet the geostrophic wind relation is much less satisfactory than at higher levels, owing to the influence of surface friction. In order to study the dynamical effects determining the vertical distribution of the wind in this layer, the frictional terms have to be included in the hydrodynamic equation (1). The term containing the divergence of the velocity may be omitted because it is of less importance. If the coefficient of viscosity μ is regarded as constant, the effect of viscosity is expressed by $\mu/\rho \nabla^2\mathbf{v}$. Since the variations in the vertical direction are much larger than in the horizontal direction, it is generally sufficient to take only the term $\mu/\rho \partial^2\mathbf{v}/\partial z^2$ into consideration.

For the sake of simplicity, it may be assumed that the wind field is steady and that the wind is the same everywhere in the horizontal direction; furthermore that the pressure gradient is directed in the positive x-direction and that $1/\rho \partial p/\partial x$ is independent of the altitude. If, finally, the pressure gradient is replaced by the geostrophic wind velocity v_g according to equations (4), the two equations determining the vertical wind distribution in the frictional layer are

$$
\begin{aligned}
-2\Omega_z(v - v_g) &= \frac{\mu}{\rho}\frac{d^2u}{dz^2}, \\
2\Omega_z u &= \frac{\mu}{\rho}\frac{d^2(v - v_g)}{dz^2}.
\end{aligned}
\tag{8}
$$

Upon multiplying the second equation by the imaginary unit i, adding it to the first equation and introducing

$$V = u + i(v - v_g),$$

it follows that

$$\frac{d^2V}{dz^2} = i\frac{2\Omega_z\rho}{\mu}V.$$

The following boundary conditions may be assumed. With increasing altitude the function V, which represents the deviation from the geostrophic wind, should tend to zero, since the frictional effects are greatest next to the ground and decrease with the altitude and since the observations show that the wind approaches the geostrophic wind with increasing elevation. At the surface of the earth the wind velocity may be zero. This condition is not entirely satisfactory because the wind distribution in the lowest few meters next to the ground is subjected to other forces than those considered in equations (8). But even with this simple boundary condition the essential features of the wind distribution in the frictional layer, except at the very lowest levels, are well brought out. In symbols the two boundary conditions require that

$$
\begin{aligned}
V &\to 0 \quad \text{when} \quad z \to \infty \\
V &= -iv_g \quad \text{when} \quad z = 0.
\end{aligned}
$$

Hence

$$V = -iv_g e^{-az}(\cos az - i \sin az),$$

where

$$a = \sqrt{\frac{\Omega_z \rho}{\mu}}.$$

Separating the real and imaginary parts, it follows that

$$\begin{aligned} u &= -v_g e^{-az} \sin az \\ v &= v_g(1 - e^{-az} \cos az). \end{aligned} \tag{9}$$

The end points of the wind vector, when plotted in the u, v plane, lie on a spiral around the end point of the geostrophic wind vector. The actual wind direction coincides with the direction of the geostrophic wind, which has here been chosen as the y-direction, at the level where $u=0$ while v is finite. The lowest level H above the surface where u vanishes is according to the first equation (9) given by the formula

$$H = \frac{\pi}{a} = \pi\sqrt{\frac{\mu}{\Omega_z \rho}}. \tag{10}$$

This height is called the "geostrophic wind level" or, less accurately, "gradient wind level." Since $e^{-\pi} = 0.043$, the frictional deviations from the geostrophic wind above the geostrophic wind level are very small and below the limits set by the errors of observation. Since $\Omega_z = \omega \sin \phi$ where the angular velocity of the earth's rotation $\omega = 7.29 \cdot 10^{-5}$ sec^{-1}, $\rho = 1.26 \cdot 10^{-3}$ gm cm^{-3}, $\mu = 1.7 \cdot 10^{-4}$ gm cm^{-1} sec^{-1}, it would follow that at latitude 45° the geostrophic wind level is only about 1.6 m. In reality, the average value of H is about 1500 m, although it may be considerably smaller or larger. With this value for H one finds that $\mu = 150$ gm cm^{-1} sec^{-1}. This figure can of course only be regarded as a rough approximation, owing to the simplifying assumptions and to the varying atmospheric conditions. The order of magnitude may be smaller by as much as a factor 1/10. But even making allowance for these effects, the coefficient of viscosity for atmospheric motions is very much larger than the one determined in the laboratory for laminar flow. Obviously, the value of μ determined by means of the height of the geostrophic wind level is not the coefficient of molecular viscosity but the coefficient of eddy viscosity caused by the turbulence of atmospheric motions. The turbulent eddy motion brings about a transport of momentum at right angles to the direction of mean motion. The effect of this momentum transport is similar to that due to the molecular viscosity, but of far greater intensity.

6. Turbulent mass exchange. In a manner similar to the transfer of momentum atmospheric turbulence gives rise also to a transport of heat, water vapor, dust and other characteristics of the air. The transfer of heat is analogous to the

molecular conduction of heat, the transfer of water vapor analogous to the process of molecular diffusion. The mechanism responsible for this transfer is called turbulent mass exchange. Since most of the atmospheric characteristics change much more rapidly in vertical than in horizontal direction, the distribution of a property s due to turbulent mass exchange can be described by the differential equation

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial z}\left(k \frac{\partial s}{\partial z}\right), \tag{11}$$

which is similar to the equation for the conduction of heat. But while in the process of molecular transfer the coefficient k can, as a rule, be regarded as constant, such is not the case when turbulent mass exchange is considered, and the assumption of a constant k (or μ as in the preceding section) represents only a rough approximation to reality. Since equations of the type (11) describe problems involving atmospheric turbulence, all the mathematical methods used in the theory of the conduction of heat may find application in dynamic meteorology also.

One of the standard examples concerns the daily temperature period. The air temperature at the earth's surface has a period of 24 hours, due to the daily period of the insolation, with a maximum about two hours after noon and a minimum at sunrise. According to the observations, the amplitude of the daily temperature wave decreases and the maximum occurs later with increasing altitude. The solution of the differential equation (11) for turbulent mass exchange shows that both these phenomena are due to the role of turbulent mass exchange in propagating the daily temperature wave upwards.

Another problem involving turbulent mass exchange arises in connection with the heating of cold continental air masses from the ground upwards as they move over increasingly warmer land. Such heating takes place when air from northern Canada and Alaska travels southward. Eq. (11) has then to be solved under a certain initial condition, depending on the original vertical temperature distribution in the air and under a boundary condition given by the temperature at the earth's surface with which the air is in contact. The solution is an expression which contains the error function.

7. Conclusion. In the foregoing discussion, nothing has been said about more advanced topics, such as the perturbation theory of atmospheric motions, because under the present emergency training program these subjects cannot be included in the curriculum. But the few examples presented here will show that in meteorology, as well as in physics, at least a knowledge of differential and integral calculus and of the more elementary parts of the differential equations of mathematical physics is highly desirable, even for the meteorology student who intends to specialize in weather forecasting. These subjects are rather minimum requirements, and there is no scarcity of more difficult problems in meteorology whose solutions require a considerably more advanced and complete mastery of mathematics.

DIAGNOSTIC TESTING PROGRAM IN PURDUE UNIVERSITY

3. A Report of the Results of the Program

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In general, universities attempting to remedy the deficiencies of beginning students in mathematics have separated the poorly prepared and given them extra instruction. This sectioning of the students has usually been determined by the number of units of high school algebra the student has had, by his score on some prognostic test, by a "weeding out" on the basis of teachers' judgments of the obviously weak during the first few weeks in the regular college course, or by some combination of these factors. In most cases this extra instruction has consisted in starting at the first part of an algebra textbook—rather than with some later chapter—and devoting more time to the study of algebra.

In those colleges and universities where the students study trigonometry first and algebra second the need for remedial instruction in the prerequisite algebra is most apparent. It is difficult, however, to select those topics and concepts in algebra which can be reviewed quickly with most benefit to the students. Some institutions have made an *a priori* determination of the material to be included in the review which has been satisfactory for them. Very few have attempted to determine objectively by an analysis of preliminary diagnostic tests exactly what topics and concepts in elementary algebra are actually causing the most trouble and the characteristic kinds of mistakes which are made so that a definite effort could be made to correct these specific difficulties in the review instruction.

Prior to June, 1942 the freshmen engineering students enrolled in mathematics were not sectioned according to abilities. The curriculum provided that the first two-thirds of the first semester be devoted to the study of trigonometry and the remainder of the semester to certain more advanced topics in algebra. With this course of study, the student who could not perform the fundamental prerequisite algebraic manipulations and numerical computations with a reasonable degree of mastery was definitely lost before he started the trigonometry.

One of the objects of the diagnostic testing program at Purdue University was to attempt to correct this situation:

(1) By an analysis of the problems in the textbook to find out specifically those mathematical skills and concepts that are needed by the student.

(2) By giving diagnostic tests to determine the abilities of entering students to perform these necessary operations before remedial work was undertaken.

(3) By remedial instruction aimed at correcting the specific weaknesses revealed by the diagnostic tests both as to the kind and type of error.

The work of the first year in our research program was experimental exploration in a field where relatively little is known. Preliminary diagnostic and achievement tests were administered. Error analyses were made and the problems were validated and revised where necessary.*

* Reports of the error analyses of the diagnostic tests were published in the papers listed at the end of this article.

The students who were tested during the second year were used as the control group and those taught in the third year as the experimental group. Evidence for the comparability of the experimental and control groups has been presented below. Remedial instruction with the control group was given incidentally as would ordinarily be done by many instructors in trigonometry. However, the specific knowledge of students' difficulties acquired by detailed analysis of the diagnostic and achievement tests may have influenced the remedial instruction as well as the presentation of new material. If so, this would only serve to minimize the observable differences between the experimental and control groups resulting from the remedial instruction given the students the third year. Otherwise the course followed the regular prescribed outline.

The achievement of the experimental group was measured by use of the same objective tests as those used for measuring the achievement of the control group. The tests were scored both years on the same basis. To insure uniformity in the scoring, the test items were graded on a right or wrong basis. There was no partial credit given where only trivial errors were made except on two tests. These two tests were almost entirely tests of computational ability for which the partial scoring was definitely made uniform for both groups.

The same diagnostic tests, psychological examination, and mathematics training test were given both groups at the beginning of the semester. All measurements of the initial abilities of the two groups indicate that the control group was on the average somewhat superior. The data of initial abilities is given in Table 1.

Immediately after the control group had been given the diagnostic tests they started the study of trigonometry. The experimental group, however, was given seven periods of remedial instruction in algebra first. The topics reviewed were (1) factoring, (2) simplification, multiplication, and division of simple fractions, (3) addition and subtraction of fractions, (4) positive integers, negative integers, and zero used as an exponent, (5) fractional exponents and radicals, (6) linear equations and systems of linear equations, and (7) quadratic equations. The material to be included in this review was determined in the manner previously outlined. A definite effort was made to correct the specific types of errors which the analysis of the diagnostic tests revealed were most common. After this review the students were given tests equivalent in difficulty to the preliminary tests and covering the same fundamental operations. The mean gain on the tests given after the review was approximately forty per cent using the initial mean as the base.

There was no attempt made in this experiment to have the instructors use the same teaching methods. In teaching the experimental group it was agreed to emphasize certain concepts and to attempt to correct certain types of mistakes. Each instructor, however, presented the material in the manner which he believed was most effective for him.

After this short review of algebra the experimental group studied exactly the same topics and in the same order as the control group. As each new topic

TABLE I

Summary of Data on Initial Comparability of the Experimental and Control Groups

		N	Mean	*S.D.*	$S.E._M$	$S.E._\sigma$
Psychological	1940	131	59.40	25.00	2.18	1.54
Examination	1941	139	53.75	26.70	2.26	1.60
Diff.			5.65±3.14	1.70±2.22		
C.R.			1.80	.76		
Significant at*			5% level	not significant		
Iowa Mathematics	1940	128	59.25	23.50	2.08	1.47
Training Test	1941	134	58.40	26.55	2.29	1.62
Diff.			.85±3.09	3.05±2.19		
C.R.			.27	1.39		
Significant at			not significant	not significant		
K-S-R	1940	130	17.78	6.50	.57	.40
Number Technique	1941	138	15.44	6.62	.56	.40
Test						
Diff.			2.34± .81	.12± .56		
C.R.			2.89	.21		
Significant at			1% level	not significant		
Purdue Mathematics	1940	130	16.72	6.70	.59	.41
Training Test	1941	136	15.10	6.60	.57	.40
Diff.			1.62± .82	.10± .57		
C.R.			1.97	.17		
Significant at			5% level	not significant		

* As is customary, the fiducial limits of significance have been here set at 95 per cent as an acceptable fiducial probability. See Peters, C. C. and Van Voorhis, Statistical Procedures and Their Mathematical Bases, McGraw Hill, 1940, pp. 137 ff.

was presented to the experimental group a definite effort was made to correct those mistakes and wrong concepts which an error analysis of the control group's achievement tests had shown to be common. Thus the emphasis was placed on those errors which the students actually make and not on what the instructors might, on other grounds, have thought needed emphasis.

The findings and grade distributions for each of the eight achievement tests in trigonometry for both groups are given in Table 2. The content of each test is briefly indicated:

Test 1. 44 Problems. Definitions of the trigonometric functions; trigonometric functions of angles of any magnitude in standard position; complementary angles; trigonometric functions of angles of 0°, 30°, 45°, 60°, 90°, 120°, 135°, etc.

Test 2. 18 Problems. Approximate numbers; 3-place tables of trigonometric functions; solutions of right triangles.

Test 3. 40 Problems. Reduction formulas; trigonometric functions of angles of any magnitude from 3-place tables.

Test 4. 6 Problems. Numerical computations using Law of Sines and Law of Cosines, without logarithms.

Test 5. 12 Problems. Logarithms; Law of Sines by use of logarithms.

Test 6. 4 Problems. Solutions by Law of Tangents and half-angle formulas, using logarithms.

Test 7. 26 Problems. Radian measure; trigonometric equations; simple identities.

Test 8. 18 Problems. Addition formulas; multiple-angle formulas; inverse functions.

TABLE II

Data for the Eight Achievement Tests in Trigonometry

Test	Group	N	Mean	%A	%B	%C	%D	Pass	Instruction time in days	$\frac{M_1-M_2}{\sigma_{M_1-M_2}}$	In favor of	Significance level
1	Con.	138	26.6	13	32	24	31	69.3	6	2.9	Exp.	1%
	Exp.	142	30.2	24	32	23	21	78.8	6			
2	Con.	137	12.4	27	29	20	24	75.9	5	2.4	Exp.	1%
	Exp.	141	13.5	33	36	15	16	83.7	4			
3	Con.	138	34.1	63	14	12	11	89.1	3	3.9	Exp.	1%
	Exp.	143	37.6	83	10	1	6	94.4	2			
4	Con.	136	4.2	29	18	24	28	72.1	4	0.0	Neither	not sig.
	Exp.	142	4.2	25	28	20	27	72.6	3			
5	Con.	134	6.8	20	24	21	35	64.9	8	0.0	Neither	not sig.
	Exp.	140	6.8	20	33	20	27	72.8	5			
6	Con.	133	2.0	14	29	17	40	60.2	4	0.0	Neither	not sig.
	Exp.	141	2.0	18	21	21	40	60.2	4			
7	Con.	131	12.8	16	28	21	35	64.9	7	2.2	Exp.	1%
	Exp.	134	14.8	28	27	20	25	75.4	7			
8	Con.	131	9.5	14	10	30	47	53.4	8	1.8	Exp.	5%
	Exp.	135	10.7	19	20	21	40	60.0	8			

Explanation of Data: (by columns of entries)

Mean: The arithmetic mean score, based on number of problems marked *correct* on "right or wrong" scoring.

S.E.$_M$ Standard error of the arithmetic mean.

%A: The per cent of grades of "A," the highest passing letter mark.

%B: The per cent of grades of "B," the second passing mark.

%C: The per cent of grades of "C," the lowest passing mark.

%D: The per cent of grades of "D," the failing mark.

Pass: The per cent of passing grades, to the nearest 0.1%.

Days: The number of days (50 minutes per day) of instruction devoted to the subject matter for each test.

$\frac{M_1 - M_2}{\sigma_{M_1 - M_2}}$: Ratio of difference of means to its standard error.

The observable effect of the remedial instruction in the distribution of the grade averages for the whole course in trigonometry is perhaps of greater interest than any observations of effects in single tests. The distribution of grades for trigonometry is now indicated:

TABLE III

Distribution of Semester Grades in Trigonometry

	%*A*	%*B*	%*C*	% Pass	%*D*
Control	14.6	34.6	25.4	74.6	25.4
Experimental	21.9	35.7	31.4	89.0	11.0

The critical ratio of the mean difference is 2.3, which is significant at the 1% level. All tests were converted to letter grade equivalents and summed, thus weighting them equally for the determination of the semester grades. This table shows a net gain by the experimental group of fifty per cent in "*A*" grades, of seventeen per cent in "*A*" and "*B*" grades combined, and of nineteen per cent in all passing grades, with a reduction of more than fifty-six per cent in failures.

Since the teaching personnel changed during the second and third years, this paper gives only the results for those classes taught by Dr. Keller and Dr. Shreve each of whom taught three classes each year. The inclusion of the classes of the other instructors would show results even more favorable for the experiment.

Through all these tests, there are two rather obvious trends: (1) In those tests where manipulations of algebraic symbols occur frequently, the increase in mean score, per cent of passing grades, per cent of "*A*'s," and the decrease in per cent of failures are significantly high. (2) In those tests which are largely computational, the differences in results are not statistically significant. The reduction in time for teaching some of these topics, however, ranges up to 37 per cent below the time given to the same topics with the control group.

The results are equally significant if the work in algebra is included, even though there were eleven less days the third year to teach the trigonometry and algebra. This difference in time was due to the seven days used for review and a four day shorter semester.

The data warrant the conclusion that remedial instruction is of positive value when:

(1) Specific weaknesses of students are first determined by diagnostic tests,
(2) Remedial instruction in regular lesson assignments is based specifically upon the diagnostic findings, and

(3) In the teaching of each new topic those concepts and operations are emphasized which the analysis of acihevement tests shows to be common errors.

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ON AN INTEGRAL TRANSFORMATION OF GENERAL CIRCUIT THEORY

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It has been stated by Carson that the behavior of a network is completely determined if either the real or the imaginary component of the complex steady state admittance is specified over the entire frequency range [1]. The same can be said of the impedance. The direct relations existing between the two components have been derived by M. Bayard [2], who has studied the impedance function of a Kirchhoff dipole consisting of resistors, capacitors and inductors. His results have been quoted by Van der Pol [3]. The same relations appear again in the theory of anomalous dispersion and absorption, where they have been obtained by R. de L. Kronig [4] and H. A. Kramers [5]. Some applications have been given afterwards by C. J. Gorter and R. de L. Kronig [6]. Anomalous dispersion and absorption is but a particular case of anomalous behavior of dielectrics. A derivation from a very general point of view covering both networks and anomalous dielectrics, of the relations mentioned here, has been given recently by B. Gross, [7] and, independently and nearly at the same time, by R. H. Cole [8]. A numerical application in the field of anomalous dielectrics has already been given by Silva and Gross [9]. Independently of their physical significance, the relations referred to above, or quite similar ones, have been considered incidentally in mathematical papers [10]. Their closer study seems justified as much from a physical as from a mathematical point of view. This study has been made by B. Levi, L. Gama and the present author [11]. Considering the numerous applications of the formula referred to above and the general technical and scientific interest in all research concerning dielectric behavior, it seems worth while to report briefly here our main results. For details we must refer to the original papers.

I

The relations in question are integrals of the type:

$$(1) \qquad A(\beta) = \frac{2}{\pi}\int_0^\infty B(\alpha)\,\frac{\beta}{\beta^2 - \alpha^2}\,d\alpha \quad \text{where} \quad -\infty < \beta < +\infty, \quad \beta \neq 0.$$

An integral of this form exists as a *principal value* for a given β, if the function $B(\alpha)$ is such that the 4 derivate numbers of B for $\alpha=\beta$ are finite, provided furthermore that the integrals $\int_0^{\beta-\delta}$ and $\int_{\beta+\delta}^{\infty}$ exist, where δ is a positive number sufficiently small, but $\neq 0$. (L. I. Gama [11]).

The kernels of (1), given by

$$K(\alpha, \beta) = \frac{\beta}{\beta^2 - \alpha^2} \tag{2}$$

are *orthogonal* in the interval 0 to ∞, *i.e.*,

$$\int_0^{\infty} K(\alpha, \beta)K(\alpha, \sigma)d\alpha = 0, \quad \text{for} \quad \beta \neq \sigma. \tag{3}$$

The *eigenfunctions* and *eigenvalues* of the integral (1) are given, respectively, by

$$\phi_n(\beta) = c(n)\beta^n, \qquad -1 < n < +1 \tag{4}$$

and

$$\lambda_n = \frac{1}{\frac{2}{\pi}\int_0^{\infty} \frac{x^n}{1 - x^2}dx}, \qquad -1 < n < +1. \tag{5}$$

For $0 > n > -1$, this expression reduces to $\lambda_n = -\tan(\pi/2n)$. All these relations are easily verified by direct substitution.

The integral (1) can still be transformed for periodic functions. If B is an even, bounded function of period l, average value (averaged over one period) B_0, and $0 < \beta < \beta_0 < 1$, then we can write (1) in the form

$$A(\beta) = \frac{1}{l}\int_0^{l} [B(\alpha) - B_0] \cot \frac{\pi}{l}(\beta - \alpha)d\alpha. \tag{6}$$

This equation has the form of Hilbert's transformation [12]. It is obtained by expanding the integral (1) in a series of integrals each of which extends over one period of the function B.

The *derivative* of (1) is given by

$$A'(\beta) = -\frac{2}{\pi}\int_0^{\infty} B'(\alpha)\frac{\alpha}{\alpha^2 - \beta^2}d\alpha. \tag{7}$$

The integral (1) becomes particularly simple if $B(\alpha)$ is $\cos \sigma\alpha$ or $\sin \sigma\alpha$. Then

$$\frac{2}{\pi}\int_0^{\infty} \cos \sigma\alpha \frac{\beta}{\beta^2 - \alpha^2}d\alpha = \sin \sigma\beta \tag{7a}$$

$$\frac{2}{\pi}\int_0^{\infty} \sin \sigma\alpha \frac{\alpha}{\alpha^2 - \beta^2}d\alpha = \cos \sigma\beta. \tag{7b}$$

These formulas can be interpreted as representations of the functions $\sin \sigma\beta$ and $\cos \sigma\beta$ by Fourier integrals.

II

The integral (1) is closely related to the theory of conjugate Fourier series and integrals.

Consider the expressions:

$$P^*(\beta) = \sum_1^\infty k_n \sin n\beta \tag{8a}$$

$$Q^*(\beta) = \sum_1^\infty k_n \cos n\beta \tag{8b}$$

and

$$P^{**}(\beta) = \int_0^\infty \phi(\sigma) \sin \sigma\beta \, d\sigma \tag{9a}$$

$$Q^{**}(\beta) = \int_0^\infty \phi(\sigma) \cos \sigma\beta \, d\sigma. \tag{9b}$$

The functions P and Q are not independent of each other. Suppose that

$$\sum_1^\infty | k_n | \quad \text{and} \quad \int_0^\infty | \phi(\sigma) | \, d\sigma$$

exist, the series (8) and integrals (9) being in consequence absolutely and uniformly convergent.

Under these conditions the relations between P and Q are given by the formulas.

$$P(\beta) = \frac{2}{\pi} \int_0^\infty Q(\alpha) \frac{\beta}{\beta^2 - \alpha^2} \, d\alpha \tag{10a}$$

$$Q(\beta) = \frac{2}{\pi} \int_0^\infty P(\alpha) \frac{\alpha}{\alpha^2 - \beta^2} \, d\alpha. \tag{10b}$$

These relations are easily verified by direct substitution with the aid of the integrals (7).

III

The relations (10) can be considered as a pair of integral equations. There arises therefore the question as to the general conditions under which the integral (1) can be inverted.

The answer is given by the following theorem: (B. Levi [13]).

The inversion of the integral

$$A(\beta) = \frac{2}{\pi}\int_0^\infty B(\alpha)\,\frac{\beta}{\beta^2-\alpha^2}\,d\alpha \tag{11}$$

is given by

$$B(\beta) = \frac{2}{\pi}\int_0^\infty A(\alpha)\,\frac{\alpha}{\alpha^2-\beta^2}\,d\alpha \tag{12}$$

for any function $A(\beta)$, defined continuous and bounded in the interval 0 to ∞, infinitesimal to at least the first order at $\beta=0$, and such that the integrals

$$\int_0^\infty \frac{A(\beta)}{\beta}\,d\beta \quad \text{and} \quad \int_p^\infty \frac{|A(\beta)|}{\beta^2}\,d\beta, \quad \text{where} \quad p>0$$

exist.

IV

Arbitrary functions may be represented by integrals of the type (1).

Any function $F(\beta)$ which can be represented by absolutely and uniformly convergent Fourier series or integrals, may also be represented by

$$F(\beta) = \frac{2}{\pi}\int_0^\infty P(\alpha)\,\frac{\beta}{\beta^2-\alpha^2}\,d\alpha + \frac{2}{\pi}\int_0^\infty Q(\alpha)\,\frac{\alpha}{\alpha^2-\beta^2}\,d\alpha \tag{13}$$

where

$$P(\alpha) = \frac{1}{\pi}\int_{-\infty}^\infty F(\beta)\,\frac{\beta}{\beta^2-\alpha^2}\,d\beta \tag{14a}$$

$$Q(\alpha) = \frac{1}{\pi}\int_{-\infty}^\infty F(\beta)\,\frac{\alpha}{\alpha^2-\beta^2}\,d\beta \tag{14b}$$

provided, furthermore, that $\int_{-\infty}^{\infty} F(\beta)d\beta$ exists if the function is non-periodic, or that $\int_0^l F(\beta)d\beta=0$, if the function is periodic, of period l.

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CONVERGENT SEQUENCES OF PROBABILITY DISTRIBUTIONS

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1. Introduction. It is known that many of the more important probability distributions encountered in practical statistics tend to simple limiting forms as certain parameters become infinite. However, because of the lack of an adequate textbook on statistical theory at the intermediate and advanced levels, the teacher interested in the derivation of such convergence theorems is forced to refer to a literature which is widely scattered and often mathematically unsatisfactory. It is the main purpose of this paper to present a collection of some of the more important limit theorems of practical statistics, derived by a simple and uniform method of broad applicability.

The method is based upon the properties of the moment generating function. The use of this function in place of the more usual characteristic function permits us to avoid complex variables without losing certain of the advantages of the characteristic function type of proof. In an earlier expository paper [6, p. 75], the author has set forth reasons why the uniqueness and limit theorems for the moment generating function* might well be assumed without proof in certain courses in statistics. The reader who adopts this point of view will find that the present paper essentially presupposes no mathematical equipment beyond that supplied by the usual first course in calculus.

2. The m.g.f. and the s.g.f. of a distribution. Let X be a variate in one dimensional space. We shall denote the probability associated with an interval such as $a \leqq X \leqq b$ by $P(a \leqq X \leqq b)$. The function $F(x) = P(-\infty < X \leqq x)$ will be called the distribution function of X, a term which we shall generally abbreviate to d.f. The mean value, or expected value, of any function $f(X)$ will be denoted, if it exists, by the usual symbol $E[f(X)]$. We write $\mu = E(X)$, $\sigma^2 = E[(X-\mu)^2]$, and call these quantities the mean and variance of X respectively.

The moment generating function of X, henceforth to be abbreviated to m.g.f., is the function $G(\alpha) = E(e^{\alpha X})$, where α is real. Unless statement is made to the contrary, we shall always impose the restriction upon the distribution of X that this mean value is to exist for all values of α in some neighborhood of the origin. The function $H(\alpha) = \log_e G(\alpha)$ is called the semi-invariant generating function or cumulant generating function of X, to be abbreviated to s.g.f. In the earlier paper referred to above, the author has given a fairly detailed exposition of the purely formal properties of the m.g.f. and the s.g.f. It suffices for the present purposes merely to recall that $\mu = H'(0)$, and $\sigma^2 = H''(0)$. In addition, we shall need the following limit theorem:

THEOREM 2.1. *Let $F_n(x)$ and $G_n(\alpha)$ be respectively the d.f. and m.g.f. of a variate X_n and let $F(x)$ and $G(\alpha)$ be the d.f. and m.f.g. of a variate X. If $\lim_{n\to\infty} G_n(\alpha) = G(\alpha)$ for α in some neighborhood of $\alpha = 0$, then $\lim_{n\to\infty} F_n(x) = F(x)$ uniformly in each finite or infinite interval of continuity of $F(x)$.*

* Theorem 2.1 below is of this type.

A formal proof of this theorem has been supplied elsewhere by the author,* and will be omitted here. We repeat our earlier suggestion that in certain courses in statistical theory, this theorem might not inappropriately be assumed without proof.

3. A convergence test for distributions. If $G(\alpha)$ exists for $|\alpha| < \alpha_1 \neq 0$, then it is easily seen that $H(\alpha)$, $H'(\alpha)$, $H''(\alpha)$, and $H'''(\alpha)$ all exist for $|\alpha| < \alpha_1$. Accordingly we can expand $H(\alpha)$ in a Maclaurin series with remainder in the following two ways:

$$H(\alpha) = H(0) + \alpha \cdot \mu + \frac{\alpha^2}{2} \cdot H''(\theta\alpha), \qquad 0 < \theta < 1, \ |\alpha| < \alpha_1,$$

$$H(\alpha) = H(0) + \alpha \cdot \mu + \frac{\alpha^2}{2} \cdot \sigma^2 + \frac{\alpha^3}{6} H'''(\theta'\alpha), \qquad 0 < \theta' < 1, \ |\alpha| < \alpha_1.$$

The first term in each expansion vanishes, since $H(0) = \log E(e^0)$.

Suppose now that we are dealing with a variate for which $\mu = 0$ and $\sigma = 1$. Such a variate is often called a standardized or reduced variate. In this case our expansions become

$$H(\alpha) = \frac{\alpha^2}{2} H''(\theta\alpha), \qquad 0 < \theta < 1, \ |\alpha| < \alpha_1, \tag{3.1a}$$

$$H(\alpha) = \frac{\alpha^2}{2} + \frac{\alpha^3}{6} H'''(\theta'\alpha), \qquad 0 < \theta' < 1, \ |\alpha| < \alpha_1. \tag{3.1b}$$

If now X is any variate with a m.g.f., then the variate $T = (X - \mu)/\sigma$ is clearly a reduced variate. If we denote the m.g.f. and s.g.f. of T by ${}_TG(\alpha)$, ${}_TH(\alpha)$, and the s.g.f. of X by $H(\alpha)$, then

$$\begin{aligned} {}_TH(\alpha) &= \log ({}_TG(\alpha)) \\ &= \log E\left[\exp \alpha\left(\frac{X - \mu}{\sigma}\right)\right] = -\mu \frac{\alpha}{\sigma} + \log E\left[\exp \frac{\alpha X}{\sigma}\right] \\ &= -\mu \frac{\alpha}{\sigma} + H\left(\frac{\alpha}{\sigma}\right). \end{aligned}$$

Therefore ${}_TH''(\alpha) = H''(\alpha/\sigma)/\sigma^2$, ${}_TH'''(\alpha) = H'''(\alpha/\sigma)/\sigma^3$. Substituting into (3.1a) and (3.1b), we obtain

$${}_TH(\alpha) = \frac{\alpha^2}{2} \frac{H''(\theta\alpha/\sigma)}{\sigma^2}, \qquad 0 < \theta < 1, \ |\alpha| < \alpha_1,$$

$${}_TH(\alpha) = \frac{\alpha^2}{2} + \frac{\alpha^3}{6} \frac{H'''(\theta'\alpha/\sigma)}{\sigma^3}, \qquad 0 < \theta' < 1, \ |\alpha| < \alpha_1,$$

or

* See [7], where a slightly more general form of the result is stated and proved.

$$(3.2a) \qquad {}_1G(\alpha) = \exp\left\{\frac{\alpha^2}{2}\,\frac{H''(\theta\alpha/\sigma)}{\sigma^2}\right\}, \qquad 0 < \theta < 1,\ |\alpha| < \alpha_1,$$

$$(3.2b) \qquad {}_1G(\alpha) = \exp\left(\frac{\alpha^2}{2}\right)\exp\left\{\frac{\alpha^3}{6}\,\frac{H'''(\theta'\alpha/\sigma)}{\sigma^3}\right\}, \qquad 0 < \theta' < 1,\ |\alpha| < \alpha_1.$$

Now consider the reduced Gaussian or normal distribution, whose d.f. is

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}dt.$$

(In the sequel, the symbol $N(x)$ will always refer to this d.f.) A simple integration shows that the m.g.f. of the distribution is $\exp(\alpha^2/2)$. Thus (3.2a) and (3.2b) suggest that in many cases the m.g.f. of a reduced variate will be approximately equal to that of a normal distribution, at least for values of α sufficiently near the origin. These equations, when taken with Theorem 2.1, yield the following:

CONVERGENCE TEST FOR DISTRIBUTIONS. *Let $H_n(\alpha)$ be the s.g.f., μ_n the mean and σ_n^2 the variance of a variate X_n dependent on a parameter n, and let $F_n(t)$ be the d.f. of the reduced variate $T_n = (X_n - \mu_n)/\sigma_n$. If either of the equations*

$$(3.3a) \qquad \lim_{n\to\infty} \frac{H_n''(\alpha/\sigma_n)}{\sigma_n^2} = 1$$

$$(3.3b) \qquad \lim_{n\to\infty} \frac{H_n'''(\alpha/\sigma_n)}{\sigma_n^3} = 0$$

holds uniformly in some neighborhood of $\alpha = 0$, then $\lim_{n\to\infty} F_n(t) = N(t)$ uniformly for all t.

It is possible to show that (3.3a) and (3.3b) are equivalent conditions, but we shall not go into the matter here. We shall now apply our convergence test to a number of important distributions of variable form.

4. Generalized binomial distributions. The variate X_n is said to have the binomial distribution of Poisson with parameters $p_1, p_2, \cdots, p_n$, if X_n can be expressed as the sum of n independent variates $z_1, z_2, \cdots, z_n$ where z_j has the values 0 and 1 assumed with respective probabilities q_j and p_j $(q_j + p_j = 1)$. The variate X_n may be physically interpreted as the number of successes in n independent trials with probability p_j of success in the jth one. The reader will have no trouble in checking the following computation, in which $G_n(\alpha)$, $H_n(\alpha)$, μ_n, and σ_n^2 denote respectively the m.g.f., s.g.f., mean, and variance of X_n:

$$G_n(\alpha) = E\left(\exp \sum_1^n \alpha z_j\right) = \prod_1^n (q_j + p_j e^{\alpha}),$$

$$H_n(\alpha) = \sum_1^n \log (q_j + p_j e^{\alpha}),$$

$$\mu_n = \sum_1^n p_j = n\bar{p}, \qquad \bar{p} = \frac{\sum_1^n p_j}{n},$$

$$\sigma_n^2 = \sum_1^n q_j p_j = n\overline{pq} - n\sigma_p^2, \qquad \bar{q} = 1 - \bar{p},\ \sigma_p^2 = \frac{\sum_1^n (p_j - \bar{p})^2}{n},$$

$$H'''(\alpha) = e^\alpha \sum_1^n q_j p_j \frac{q_j - p_j e^\alpha}{(q_j + p_j e^\alpha)^3}.$$

So

$$|H_n'''(\alpha)| \leqq e^\alpha \sum_1^n \frac{q_j p_j}{(q_j + p_j e^\alpha)^2} \leqq e^{|\alpha|}\sigma_n^2. \tag{4.1}$$

Consider now a sequence $X_1, X_2, X_3, \cdots$, of variates with Poisson binomial distributions. Let the parameters of X_n be $p_1^{(n)}, p_2^{(n)}, \cdots, p_n^{(n)}$, the variance be σ_n^2, and the s.g.f., $H_n(\alpha)$. It is apparent from (4.1) that if $\lim_{n\to\infty}\sum_1^n p_j^{(n)} q_j^{(n)} = \lim_{n\to\infty}\sigma_n^2 = \infty$, where $q_j^{(n)} = 1 - p_j^{(n)}$, then $\lim_{n\to\infty} H_n'''(\alpha/\sigma_n)/\sigma_n^3 = 0$ uniformly in any finite interval. Referring to the Convergence Test of §3, we obtain the following result:

THEOREM 4.1. *Let X_n have a Poisson binomial distribution with parameters $p_1, p_2, \cdots, p_n$, which may depend upon n, and let $F_n(t)$ be the d.f. of the variate $T_n = (X_n - \sum_1^n p_j)/\sum_1^n p_j q_j$, where $q_j = 1 - p_j$. If $\lim_{n\to\infty}\sum_1^n p_j q_j = \infty$, then $\lim_{n\to\infty} F_n(t) = N(t)$ uniformly.**

If $p_j = p$, $j = 1, \cdots, n$, the distribution of X_n is a simple binomial or Bernoulli distribution. If in this case p depends upon n, the condition $\lim_{n\to\infty}\sum_1^n p_j q_j = \infty$ becomes $\lim_{n\to\infty} npq = \infty$. This condition is automatically satisfied if p is constant for all n, $0 < p < 1$, and the theorem then reduces to the familiar De-Moivre-Laplace Theorem.

The variate X_n in the theorem assumes only integral values, and so the values of T_n are equally spaced at intervals of $1/\sigma_n$. If the histogram for T_n and the graph of the reduced normal density function $N'(t)$ are sketched upon the same coordinate axes, the picture will suggest that for a given value of n, the normal distribution will yield a better approximation to $P(k \leqq X_n \leqq l)$, where k and l are integers, if we add on an interval of length $1/(2\sigma_n)$ at each end of the corresponding T_n interval. Numerical checks indicate that the approximation is indeed improved by this device.

5. The Poisson exponential distribution. A variate X is said to have a Poisson exponential distribution with parameter m if its values are $0, 1, 2, \cdots$, and if $P(X = k) = m^k e^{-m}/k!$, $k = 0, 1, 2, \cdots$. It is easily seen [6, p. 379] that $H(\alpha) = m(e^\alpha - 1)$.

* The m.g.f. of T_n is $\pi_1^n[q_j \exp(-\alpha p_j/\sqrt{\sum p_i q_i}) + p_j \exp(\alpha q_j/\sqrt{\sum p_i q_i})]$, and we have proved that this expression approaches $\exp(\alpha^2/2)$ as $n \to \infty$, provided that $\sum_1^n p_j q_j \to \infty$. This aspect of the theorem is possibly of some interest for itself alone, in that it furnishes a generalized solution to an elementary problem proposed by J. F. Kenney in this MONTHLY, vol. 49, 1942, p. 61, problem E504.

The Poisson exponential distribution is the limit of a generalized binomial distribution under certain conditions on the parameters p_j. For by expanding $\log(q_j+p_ju)$ in a Taylor series with remainder about $u=1$, replacing u by e^α and summing from 1 to n, we obtain

$$H_n(\alpha) = \sum_1^n \log(q_j + p_je^\alpha) = (e^\alpha - 1)\cdot \sum_1^n p_j - \frac{(e^\alpha-1)^2}{2}\cdot \sum_1^n \frac{p_j^2}{(q_j+p_j\xi_j)^2},$$

where ξ_j lies between 1 and e^α. If $\alpha>\log(1/2)$, then $\xi_j>1/2$, $q_j+p_j\xi_j>q_j+p_j/2 = 1-p_j/2\geqq 1/2$, and $\sum_1^n[p_j/(q_j+p_j\xi_j)]^2\leqq 4\sum_1^n p_j^2$. Thus if $\lim_{n\to\infty}\sum_1^n p_j=m$ while $\lim_{n\to\infty}\sum_1^n p_j^2=0$, then $\lim_{n\to\infty}H_n(\alpha)=H(\alpha)$ in some neighborhood of the origin, where $H(\alpha)$ is the s.g.f. in the first paragraph of this section. From Theorem 2.1 we obtain the following result:

THEOREM 5.1. *Let X_n have a Poisson binomial distribution with parameters $p_1^{(n)}, p_2^{(n)}, \cdots, p_n^{(n)}$ which depend upon n, and let $F_n(x)$ be the d.f. of X_n. If $\lim_{n\to\infty}\sum_1^n p_j^{(n)}=m$ while $\lim_{n\to\infty}\sum_1^n (p_j^{(n)})^2=0$, then $\lim_{n\to\infty}F_n(x)=M(x)$, where $M(x)$ is the d.f. of a Poisson exponential distribution with parameter m.*

The degree of approximation is discussed by Uspensky [13, pp. 135–137] in the case in which $p_j^{(n)}=p^{(n)}, j=1, \cdots, n$.

From the formula $H(\alpha)=m(e^\alpha-1)$, we find that for a Poisson exponential distribution $\mu=m$, $\sigma^2=m$, $H''(\alpha)=me^\alpha$, $H''(\alpha/\sigma)/\sigma^2=\exp(\alpha/\sqrt{m})$. Since the last expression tends uniformly to one in any finite interval as $m\to\infty$, our Convergence Test applies to this distribution.

THEOREM 5.2. *If X has a Poisson exponential distribution with parameter m, and if $F_m(t)$ is the d.f. of the variable $(X-m)/\sqrt{m}$, then $\lim_{m\to\infty}F_m(t)=N(t)$ uniformly.*

The remarks made at the end of §4 concerning the correction to the range of the variable T_n of Theorem 4.1 apply equally well in the present case to T_m. The correction here has the value $1/(2\sqrt{m})$.

6. The Γ-distribution. If the variate X has the d.f.

$$F(x) = \begin{cases} \displaystyle\int_0^x \frac{h^b}{\Gamma(b)} t^{b-1}e^{-ht}dt, & x \geqq 0, \\ 0, & x < 0, \end{cases}$$

with $b>0$, $h>0$, we shall say that X has a Γ-distribution with parameters b and h. Here

$$G(\alpha) = \int_0^\infty \frac{h^b}{\Gamma(b)} t^{b-1}e^{(\alpha-h)t}dt = \left(1-\frac{\alpha}{h}\right)^{-b}, \qquad |\alpha| < h,$$

$$H(\alpha) = b\log h - b\log(h-\alpha)$$

$$\mu = b/h, \qquad \sigma^2 = b/h^2,$$

$$H''(\alpha/\sigma)/\sigma^2 = (1 - \alpha/\sqrt{b})^{-2}.$$

The last expression obviously approaches unity uniformly in some neighborhood of $\alpha = 0$, as $b \to \infty$. We have:

THEOREM 6.1. *If X has a Γ-distribution with parameters b and h, and if $F_b(t)$ is the d.f. of the variable $T_b = h(X - b/h)/\sqrt{b}$, then $\lim_{b\to\infty} F_b(t) = N(t)$ uniformly.*

The important case is that in which $h = 1/2$, $b = n/2$, where n is a positive integer. The distribution is then usually termed a χ^2 distribution with n degrees of freedom. In this case we shall denote the variate X by χ_n^2, and the theorem states that for large values of n, the distribution of $(\chi_n^2 - n)/\sqrt{2n}$ is approximately normal.

Various tables of the distribution of χ_n^2 for low values of n are available. Perhaps the most convenient one for practical purposes is due to R. A. Fisher [8, pp. 118–119], [9]; this table gives for values of n from 1 to 30 the various points $a_n(p)$ such that $P[\chi_n^2 > a_n(p)] = p$, where p is one of several chosen probability levels, as for instance, .05. The table has recently been extended by Thompson [12].

Fisher has pointed out that $\sqrt{2\chi_n^2} - \sqrt{2n-1}$ has approximately a reduced normal distribution for $n > 30$. Wilson and Hilferty [15] showed that rather accurate results could be obtained by taking $(\chi_n^2/n)^{1/3}$ to be normally distributed with mean $1 - 2/(9n)$ and standard deviation $\sqrt{2/(9n)}$. From the fact that $(\chi_n^2 - n)/\sqrt{2n}$ is approximately normally distributed for large values of n, we obtain the approximate formula

$$a_n(p) \doteq n + a(p)\sqrt{2n}, \tag{6.1}$$

where $a(p)$ is the point of the reduced normal distribution corresponding to the probability level p. It is possible to show that actually

$$a_n(p) = n + a(p)\sqrt{2n} + \tfrac{2}{3}\{[a(p)]^2 - 1\} + 0(n^{-1/2}).^* \tag{6.2}$$

so we should not expect to obtain entirely accurate results from (6.1) even for large values of n. Fisher's suggestion yields the approximation

$$a_n(p) \doteq \tfrac{1}{2}[a(p) + \sqrt{2n-1}]^2 = n + a(p)\sqrt{2n} + \frac{[a(p)]^2 - 1}{2} + 0(n^{-1/2}). \tag{6.3}$$

From Wilson and Hilferty's approximation, we obtain

$$\begin{aligned} a_n(p) &\doteq n\left[1 - \frac{2}{9n} + a(p)\sqrt{\frac{2}{9n}}\right]^3 \\ &= n + a(p)\sqrt{2n} + \tfrac{2}{3}\{[a(p)]^2 - 1\} + 0(n^{-1/2}). \end{aligned} \tag{6.4}$$

* A. M. Peiser is preparing a paper dealing from this point of view with approximate formulas for percentage points of various distributions, including the χ^2 distribution.

In the light of (6.2), we note that (6.4) should be the best of the three approximations for large values of n. Numerical comparisons of the results given by (6.3) and (6.4) with the corresponding true points $a_n(p)$ were made by Wilson and Hilferty [15] for values of n from 1 to 30 and by Merrington [11] for values of n from 30 to 100. These comparisons consistently affirm the superiority of (6.4); Merrington found a maximum absolute error of less than .04 between the values given by (6.4) and the true values, for the values of n and p which she considered. The accuracy of (6.4) is especially remarkable for $p=.05$.*

7. The Fisher z-distribution. The variate $Z=[\log(\chi_m^2/m)-\log(\chi_n^2/n)]/2$, where χ_m^2 and χ_n^2 have independent χ^2 distributions with respectively m and n degrees of freedom, is said to have the Fisher z-distribution. The d.f. of Z is

$$\int_{-\infty}^{z} \frac{2m^{m/2}n^{n/2}}{B\left(\frac{m}{2},\frac{n}{2}\right)} \frac{e^{mt}}{(me^{2t}+n)^{(n+m)/2}}\,dt,$$

where $B(x, y)$ is the Beta function. The m.g.f. $G(\alpha)$ is perhaps most easily obtained by referring the integration back to the two χ^2 distributions:

$$\begin{aligned} G(\alpha) &= E(e^{\alpha Z}) \\ &= E\left\{\exp\left[\frac{\alpha}{2}\left(\log\frac{\chi_m^2}{m}-\log\frac{\chi_n^2}{n}\right)\right]\right\} \\ &= \left(\frac{n}{m}\right)^{\alpha/2} E[(\chi_m^2)^{\alpha/2}]E[(\chi_n^2)^{-\alpha/2}] \\ &= \left(\frac{n}{m}\right)^{\alpha/2} \frac{\Gamma\left(\frac{m+\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}, \qquad -m<\alpha<n. \end{aligned}$$

Then if we let $\psi(u)=\log\Gamma(u)$, we have for the s.g.f.:

$$H(\alpha)=\frac{\alpha}{2}[\log n-\log m]+\psi\left(\frac{m+\alpha}{2}\right)+\psi\left(\frac{n-\alpha}{2}\right)-\psi\left(\frac{m}{2}\right)-\psi\left(\frac{n}{2}\right),$$

$$H'(\alpha)=\frac{1}{2}\left[\log n-\log m+\psi'\left(\frac{m+\alpha}{2}\right)-\psi'\left(\frac{n-\alpha}{2}\right)\right],$$

$$\mu=\frac{1}{2}\left[\log n-\log m+\psi'\left(\frac{m}{2}\right)-\psi'\left(\frac{n}{2}\right)\right],$$

* Aitken [1, p. 104] states that the 5 per cent point of the distribution of χ_n^2 (that is, $a_n(.05)$ in our notation) "is given with good approximation" by $n(2-.025n)$. The assertion is doubtless meant to apply only for small values of n, say $n<20$; but for such values of n, the formula seems to yield a better approximation to $a_{n-1}(.05)$ than to $a_n(.05)$. By the time $n=30$, both interpretations of the formula are about 12 per cent in error.

$$H''(\alpha) = \frac{1}{4}\left[\psi''\left(\frac{m+\alpha}{2}\right) + \psi''\left(\frac{n-\alpha}{2}\right)\right],$$

$$\sigma^2 = \frac{1}{4}\left[\psi''\left(\frac{m}{2}\right) + \psi''\left(\frac{n}{2}\right)\right],$$

where $-m<\alpha<n$.

From Binet's second formula for $\psi(u)$ [14, p. 251], we find that if $u>0$, then

$$\psi'(u) = \log u - \frac{1}{2u} + R_1(u), \qquad |R_1(u)| < \frac{1}{12u^2},$$

$$\psi''(u) = \frac{1}{u} + R_2(u), \qquad |R_2(u)| < \frac{2}{3u^2}.$$

Substituting into the earlier set of equations, we have

$$\mu = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{m}\right) + r_1(m, n), \tag{7.1}$$

$$H''(\alpha) = \frac{1}{2}\left(\frac{1}{m+\alpha} + \frac{1}{n-\alpha}\right) + r_2(m+\alpha, n-\alpha), \quad -m<\alpha<n. \tag{7.2}$$

$$\sigma^2 = \frac{1}{2}\left(\frac{1}{m} + \frac{1}{n}\right) + r_2(m, n), \tag{7.3}$$

where $|r_1(n, m)| < 1/(6m^2)+1/(6n^2)$, $|r_2(m, n)| < 2/(3m^2)+2/(3n^2)$. From (7.2) without using (7.3), we obtain

$$\frac{H''(\alpha/\sigma)}{\sigma^2} = \frac{1}{2}\left[\frac{m+n}{mn\sigma^2 + \alpha\sigma(n-m) - \alpha^2}\right] + \frac{r_2\left(\frac{\sigma m+\alpha}{\sigma}, \frac{\sigma n-\alpha}{\sigma}\right)}{\sigma^2}. \tag{7.4}$$

According to (7.3), $\sigma m+\alpha$ and $\sigma n-\alpha$ tend to infinity when n and m do, so the last term in (7.4) tends to zero. If the first term on the right in (7.4) be inverted and (7.3) be again applied, it will be seen at once that this term approaches unity uniformly for α in some neighborhood of the origin. Our Convergence Test, trivially extended to the case of a double sequence of m.g.f.'s, now yields the following theorem:*

THEOREM 7. 1. *Let $F_{m,n}(t)$ be the d.f. of the reduced z-distribution. Then* $\lim_{m\to\infty, n\to\infty} F_{m,n}(t) = \lim_{m\to\infty}\lim_{n\to\infty} F_{m,n}(t) = \lim_{n\to\infty}\lim_{m\to\infty} F_{m,n}(t) = N(t)$ *uniformly.*

The uniformity of the limit and the continuity of $N(t)$ imply that if $\{t_{m,n}\}$ is a sequence of points on the t-axis tending to the limit t as m and n tend to infinity in any manner, then $F_{m,n}(t_{m,n})$ tends to $N(t)$. This remark enables us to put the theorem in a more convenient form:

* A proof without the use of the m.g.f. has recently been published by Aroian 2, pp. 439–441.

THEOREM 7.2. *Let* $\mu_1=(1/n-1/m)/2$ *and* $\sigma_1=(1/m+1/n)/2$, *and let* $\mathcal{F}_{m,n}(t)$ *be the d.f. of the variate* $T=(Z-\mu_1)/\sigma_1$. *Then* $\lim_{m\to\infty,n\to\infty}\mathcal{F}_{m,n}(t)=\lim_{m\to\infty}\lim_{n\to\infty}\mathcal{F}_{m,n}(t)=\lim_{n\to\infty}\lim_{m\to\infty}\mathcal{F}_{m,n}(t)=N(t)$.

For we note that

$$P\left\{\frac{Z-\mu_1}{\sigma_1}\leqq t\right\}=P\left\{\frac{Z-\mu}{\sigma}\leqq t_{m,n}\right\},$$

where $t_{m,n}=(\sigma_1/\sigma)t+(\mu_1-\mu)/\sigma$; and from (7.1) and (7.3) we see that $t_{m,n}\to t$ as $m\to\infty$, $n\to\infty$.

A table of points of the z-distribution due to Fisher and Deming is readily available [8, pp. 250–255], [9]. The derivation of the approximate formulas for the 5 per cent and 1 per cent points recommended in connection with these tables [8, p. 239] has recently been published by Cochran [4]. It does not seem to be emphasized sufficiently in most of the texts on statistical methodology which refer to these tables that the instructions which accompany the tables have the effect of doubling the probability listed at the head of the table.*

8. The distribution of a linear form. Suppose that the n independent variates X_j each have the same s.g.f. $H(\alpha)$, $|\alpha|<\alpha_1$. Then the s.g.f. $H_n(\alpha)$ of the linear form $L_n=\sum_1^n c_jX_j$, where the numbers c_j may depend upon n, can be computed as follows:

$$H_n(\alpha)=\log E[\exp(\alpha\textstyle\sum_1^n c_jX_j)]=\log\prod_1^n E[\exp(\alpha c_jX_j)]=\sum_1^n H(\alpha c_j),$$

and $H_n(\alpha)$ exists in some neighborhood of the origin. Letting μ and σ^2 be the common mean and variance of the variates X_j, and μ_n and σ_n^2 the mean and variance of L_n, we have:

$$H_n'(\alpha)=\sum_1^n c_jH'(\alpha c_j),\qquad \mu_n=\mu\sum_1^n c_j$$

$$H_n''(\alpha)=\sum_1^n c_j^2H''(\alpha c_j),\qquad \sigma_n^2=\sigma^2\sum_1^n c_j^2,$$

$$\frac{H_n''(\alpha/\sigma_n)}{\sigma_n^2}=\frac{1}{\sigma^2}\sum_1^n d_j^2H''\left(\frac{\alpha}{\sigma}d_j\right),$$

where $d_j^2=c_j^2/\sum_1^n c_k^2$. (Notice that $\sum_1^n d_j^2=1$). Then

$$\left|\frac{H_n'(\alpha/\sigma_n)}{\sigma_n^2}-1\right|=\frac{1}{\sigma^2}\left|\sum_1^n d_j^2\left[H''\left(\frac{\alpha}{\sigma}d_j\right)-H''(0)\right]\right|$$

$$\leqq\frac{1}{\sigma^2}\left\{\max\left|H''\left(\frac{\alpha}{\sigma}d_j\right)-H''(0)\right|,\ j=1,\cdots,n\right\}.$$

It is apparent that $H_n(\alpha)$ will satisfy the Convergence Test of §3 if the quantities d_j tend to zero. Thus we have proved the following theorem:

* For further explanation of this remark, see [10, p. 144].

THEOREM 8.1. Let $L_n=\sum_1^n c_jX_j$ *be a linear form in the n identically distributed independent variates* $X_1, \cdots, X_n$, *whose m.g.f. exists in some neighborhood of the origin. Let* D_n *be an upper bound of the quantities* $(c_j)^2/\sum_1^n c_k^2$, $j=1, \cdots, n$. Let $F_n(t)$ *be the d.f. of the variate* $T_n=(L_n-\mu\sum_1^n c_j)/\sigma\sqrt{\sum_1^n c_j^2}$, *where* μ *and* σ *are the mean and variance of the variate* X_j. *Then if* $\lim_{n\to\infty} D_n=0$, *it follows that* $\lim_{n\to\infty} F_n(t)=N(t)$ *uniformly.*

The theorem is a special case of the Central Limit Theorem of probability theory [5, pp. 56 ff.]. The important cases are those in which the c_j's depend upon n. If $c_j=1/n$, $j=1, \cdots, n$, the function L_n becomes the "arithmetic mean" $\overline{X}$ of the variates X_j, and the theorem states that the distribution of $\sqrt{n}(\overline{X}-\mu)/\sigma$ tends to normality, provided that the m.g.f. of the variates X_j exists. A slight extension of the theorem serves to prove that the reduced distribution of the difference of two arithmetic means tends to normality. The two means become two proportions if the variate X_j assumes merely the values 0 and 1. The theorem may also be used to derive conditions under which the reduced distributions of the regression coefficients tend to normality.

Recently Berry [3] has obtained the following estimate of the error term in Theorem 8.1 in the more general case in which only a finite third absolute moment of the distribution of X_j is assumed:

$$\sup_{-\infty<t<\infty} |F_n(t) - N(t)| \leqq 1.88\sqrt{D_n}\,\nu_3/\sigma^3, \qquad \nu_3 = E(|X_j - \mu|^3).$$

9. The distributions of positional means. We shall indicate in this final section how Theorem 4.1 can be used to derive limit theorems for certain positional means of a sample. Specifically, we shall consider the median, but our methods are applicable at once to the other quartiles, deciles, and percentiles.

Let $X_1, X_2, \cdots, X_N$ be independent identically distributed variates.* A determination of these N variates will be called a sample. For each point $(x_1, x_2, \cdots, x_N)$ in the space of the joint distribution of the variates X_j, we define a function $w_r=f_r(x_1, x_2, \cdots, x_N)$ as follows: The numbers $x_1, x_2, \cdots, x_N$ are to be arranged in a row in order of magnitude, and w_r is the value of the rth number in this row. The corresponding variate $W_r=f_r(X_1, X_2 \cdots, X_N)$ is the *a priori* rth position in the rank list of the sample. For example, if $N=2n+1$, $r=n+1$, W_r is the *a priori* median of the sample. Henceforth we drop the phrase *a priori*, as is customary in sampling theory.

Let $H_r(w)$ denote the d.f. of W_r and $F(x)$ that of the variate X_j. A moment's reflection will show that $H_r(w)$ is equal to the probability that at least r of the variates X_j fall in the interval $-\infty<x\leqq w$. Since the probability that any one does so is $F(w)$, the law of repeated trials at once gives

$$H_r(w) = \sum_{k=r}^{N} {}_NC_k[F(w)]^k[1 - F(w)]^{N-k}.$$

* No assumption is made in this section as to the existence of a m.g.f. for the X_j's.

That is, if we were to introduce a variate Y with a simple binomial distribution with $N+1$ values and parameter $p=F(w)$, then $H_r(w)=P(r\leqq Y\leqq N)=P(r\leqq Y<\infty)$.

Now let W_r be the median of a sample of $N=2n+1$ observations; in this case, $r=n+1$ and we replace W_r by the symbol $\tilde{X}$. Let us impose upon $F(x)$ the condition that the equation $F(x)=1/2$ should have a unique solution $x=\tilde{\mu}$, which we shall call the median of the distribution of the variates X_j. In §8 we saw that the distribution of $\sqrt{2n+1}(\overline{X}-\mu)/\sigma$ becomes normal as $n\to\infty$, where μ and σ^2 are the mean and variance (assumed in that section to exist) of the variates X_j. This suggests that if we were to choose the constant K correctly, the distribution of $V_n=\sqrt{2n+1}(\tilde{X}-\tilde{\mu})/K$ might behave similarly—a conjecture which we shall now investigate.

Let $J_n(v)$ be the d.f. of V_n. Then $J_n(v)=H_{n+1}(\tilde{\mu}+Kv/\sqrt{2n+1})=P(n+1\leqq Y_n<\infty)$, where Y_n is a variate with a simple binomial distribution with $2n+2$ values and parameter $p(n)=F(\tilde{\mu}+Kv/\sqrt{2n+1})$. We now further assume that $F(x)$ is continuous at $x=\tilde{\mu}$. Then $(2n+1)p(n)[1-p(n)]\to\infty$ as $n\to\infty$; so if

$$T_n=\frac{Y_n-(2n+1)p(n)}{\sqrt{(2n+1)p(n)[1-p(n)]}},$$

then according to Theorem 4.1,

$$\lim_{n\to\infty}P(t\leqq T_n<\infty)=\int_t^\infty\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz,$$

uniformly in t. Furthermore, by the remark following Theorem 7.1, if $\{t_n\}$ is any sequence tending to t as $n\to\infty$, then $\lim_{n\to\infty}P(t_n\leqq T_n<\infty)=\lim_{n\to\infty}P(t\leqq T_n<\infty)$.

Now $J_n(v)=P(n+1\leqq Y_n<\infty)=P(t_n\leqq T_n<\infty)$, where

$$(9.1)\quad t_n=\frac{(n+1)-(2n+1)F(\tilde{\mu}+Kv/\sqrt{2n+1})}{\sqrt{(2n+1)F(\tilde{\mu}+Kv/\sqrt{2n+1})[1-F(\tilde{\mu}+Kv/\sqrt{2n+1})]}}.$$

At this stage we impose on $F(x)$ the additional (and final) restriction that $F'(\tilde{\mu})$ exists and is positive. Then

$$F(\tilde{\mu}+Kv/\sqrt{2n+1})=F(\tilde{\mu})+\frac{Kv}{\sqrt{2n+1}}[F'(\tilde{\mu})+0(1)].$$

Substituting in (9.1) and noting that $F(\tilde{\mu})=\frac{1}{2}$, we find that $\lim_{n\to\infty}t_n=-2KvF'(\tilde{\mu})$, so

$$\lim_{n\to\infty}J_n(v)=\int_{-2KvF'(\tilde{\mu})}^\infty\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz=\int_{-\infty}^{2KvF'(\tilde{\mu})}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz.$$

Obviously the proper choice for K is $K=1/[2F'(\tilde{\mu})]$. Our result may be stated formally as follows:

THEOREM 9.1. *Let $\tilde{X}$ be the median of a sample of $(2n+1)$ independent observations, each with the d.f. $F(x)$. Let $F'(\tilde{\mu})$ exist and be positive, where $\tilde{\mu}$ is the median (assumed to exist) of the distribution specified by $F(x)$. Let $J_n(v)$ be the d.f. of the variate $V_n = 2F'(\tilde{\mu})\sqrt{2n+1}(\tilde{X}-\tilde{\mu})$. Then $\lim_{n\to\infty} J_n(v) = N(v)$.*

More briefly, the median in large samples drawn by simple random sampling from a continuous parent distribution is approximately normally distributed with mean $\tilde{\mu}$ and variance $1/\{4[F'(\tilde{\mu})]^2(2n+1)\}$.

If applied to the first and third quartiles of a simple sample of $4n+1$ observations, our method yields the results that these statistics in large samples from a continuous parent distribution are approximately normally distributed with respective means q_k and variances $3/\{(64n+16)[F'(q_k)]^2\}$, $k=1, 3$, where q_1 and q_3 are the first and third quartiles of the X_j distribution.

In conclusion, it is worth pointing out that if Z has the Fisher z-distribution of §7, and if m and n are even integers, then an integration by parts will, show that $P(a \leqq e^{2Z} < \infty)$ is equal to the sum of the first $m/2$ terms, and, $P(-\infty < e^{2Z} \leqq a)$ is equal to the sum of the last $n/2$ terms, of the binomial expansion of $(q+p)^{(m+n-2)/2}$, with $p = ma/(n+ma)$, $q = 1-p$. The distribution of e^{2Z} has been tabulated by Snedecor under the name of the F distribution [9], and this provides a means of determining the points of the distribution of W_r in terms of those of $F(w)$. For example, in the case of the median of a sample of $N=23$, we find that if $H_{12}(w) = .95$, then $F(w) = .66$; that is, the 5 per cent point of the exact distribution of the median is the 34 per cent point of the distribution of the variates X_j.

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BOOLEAN ALGEBRA AS AN INTRODUCTION TO POSTULATIONAL METHODS*

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The purpose of this paper is to suggest how a simple system of postulates and theorems for Boolean algebras might be used as an introduction to postulational methods in an elementary course in foundations, or fundamental concepts, of mathematics.

Postulational theory often is introduced in such a course by the study of significant sets of postulates closely related to elementary algebra or geometry. This procedure undoubtedly is valuable. The approach suggested here, however, begins with postulates for a mathematical system quite different from the ordinary systems of elementary mathematics. At the same time, the postulates are so selected that only one of them would seem unnatural or queer to undergraduates. It is believed that there may be some psychological advantages in an approach of this kind.

The postulates. The proposed set of postulates is listed below as Set R. Postulates 1–8 are identical with a set of postulates for Boolean rings given as Set III in a previous paper.† Stone has proved‡ that any Boolean ring with unit element is equivalent to a Boolean algebra (assuming suitable connecting definitions for the two systems). Thus the inclusion of Postulate 9 actually makes Set R a set of postulates for a Boolean algebra.

We presuppose a class K of elements $a, b, c, \cdots$ and two undefined operations $+$, $\times$, which may be referred to as ring addition and Boolean multiplication respectively. Hypotheses concerning elements belonging to K are not completely stated. The postulates are referred to briefly by name in parentheses.

Set R

1. If a, b are in K then i) $a+b$, and ii) ab are in K. (Closure)
2. $(a+b)+c=a+(b+c)$ (Associative addition)
3. If $a+b_1=a+b_2$ then $b_1=b_2$ (Left-hand cancellation)
4. If $a_1+b=a_2+b$ then $a_1=a_2$ (Right-hand cancellation)
5. $(ab)c=a(bc)$ (Associative multiplication)
6. $a(b+c)=ab+ac$ (Left-hand distributive)
7. $(a+b)c=ac+bc$ (Right-hand distributive)
8. $aa=a$ (Idempotent law)
9. There exists in K an element e such that for all a in K $ae=a$. (Unit element)

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† E. R. Stabler, Sets of postulates for Boolean rings, this MONTHLY, vol. 48, 1941, pp. 20–28. (To be referred to later as [S]).

‡ M. H. Stone, The theory of representations for Boolean algebras, Transactions of the American Mathematical Society, vol. 40, 1936, pp. 37–111.

Odd and even integers. Of course, the one postulate in Set R which is not of standard type is the idempotent law, asserting that for every element a (in K) $aa=a$. Some persons may object that this postulate is impossible or contrary to the ordinary laws of algebra. This reaction immediately calls for a concrete interpretation of the abstract system which will show that all of the postulates can be simultaneously valid. Such an example will establish the logical *consistency* of the postulates. Probably the most convincing example from the standpoint of the beginner would be one in which the operations $+$ and $\times$ are closely related to ordinary addition and multiplication. Accordingly, we can present the example of addition and multiplication of odd and even integers. In particular, let us refer to two elements called "odd" and "even" and formulate the obvious tables as indicated under Interpretation I. It is easy to see that this odd-even system satisfies all of the postulates.

Interpretation I

$+$	even	odd
even	even	odd
odd	odd	even

$\times$	even	odd
even	even	even
odd	even	odd

The theorems. With the consistency of the postulates established (and the postulates thus related to some ideas of a concrete nature) the next step would probably be the abstract deduction of some consequences of the postulates. From the standpoint of abstract algebra the immediate theorems which follow are most concisely stated in the terminology of groups and rings, but it is unnecessary to use this terminology. The results can merely be worked out as a new type of algebra. The proofs are all straightforward and the rôles played by the various postulates can be clearly identified. The need of proceeding in a formal manner should be apparent in view of the novel situation caused by the idempotent law. Some of the theorems will seem equally novel. Thus it will be found, as in [S], that every element is its own inverse with respect to addition ($a+a=0$), so that in solving equations of the type $x+a=b$ it is permissible to transpose terms without changing signs. Or again, as proved by Stone, if there are more than two elements in the system then there always exist "divisors of zero," *i.e.* elements a, b, with $a\neq 0$, $b\neq 0$, but with $ab=0$. The strength of the idempotent law in accounting for these results can be readily appreciated, and further comparisons can be made with standard properties of elementary algebra.

There is a certain analogy here with the classical geometric example for illustrating apparently conflicting mathematical theorems, namely the existence of both euclidean and non-euclidean geometries. In that example, it is true, we find a more clear-cut variation of certain theorems with changes in a crucial postulate. However, difficulties are sometimes encountered in an elementary discussion due to the confusion in the minds of the students between the mathematical-logical and the physical-intuitive aspects of geometry. Furthermore, it is not easy actually to deduce significant non-euclidean theorems in a brief, logical

development. Our algebraic example seems to avoid difficulties of this kind, and may thus prepare for a later postulational study of non-euclidean geometries.

Certain theorems following from Set R will be of a familiar rather than novel nature and will look as fundamental as the postulates; for example, the commutative laws of ring addition and multiplication. Someone may suggest that these principles might have been included with the postulates. In that case they would have been redundant. Thus the question arises: are any of the nine original postulates superfluous, or are these postulates logically independent of each other? It can be understood immediately that Postulate 8, the idempotent law, is independent of the other postulates. The simplest formal example to show this would probably be the following. Let K be the class of all positive integers, with $+$ and $\times$ taken to mean ordinary addition and multiplication; then the idempotent law fails while all of the other postulates are satisfied. In the same way, examples can be found to establish the independence of most of the other postulates.*

Interpretation from logic. Next, we might inquire whether the postulates of Set R can refer to any more substantial type of subject matter than odd and even integers. At this point it would be significant to give an interpretation from logic which satisfies all of the postulates. Consider a collection of propositions (p, q, r, $\cdots$) of such a nature that they can be classified as "true" or "false." New propositions can be formed from these in various ways. For example, the *conjunction* of p and q is the proposition saying "both p and q are true"; the *disjunction* of p and q is the proposition saying "p is true or q is true, and perhaps both"; while the *complete disjunction* of p and q is the proposition saying "*either* p is true *or* q is true, but not both." Now let us interpret $p+q$ to mean the complete disjunction of p and q, and $p \times q$ to mean the conjunction of p and q. It is evident that the truth or falsity of $p+q$ and $p \times q$ is uniquely determined by the truth or falsity of the propositions p, q, themselves. Thus we can formulate truth value tables as indicated under Interpretation II, and we can think of this interpretation simply as a system of two elements "true" and "false."

Interpretation II

$+$	false	true	$\times$	false	true
false	false	true	false	false	false
true	true	false	true	false	true

On comparing the new tables with the original tables for the odd-even system, it is apparent that the new tables are simply a translation of the old ones in which the word "even" is translated as "false," and "odd" is translated as

* In [5] examples were given which proved the independence of the first eight postulates, the Boolean ring postulates. A number of the examples can be used again for Set R, but independence proofs remains to be found for Postulates 3 and 7 of Set R.

"true." Since all possible cases of the nine postulates are satisfied by the odd-even system, therefore they must be satisfied by the true-false system. Thus we have a simple illustration of two isomorphic interpretations of the same abstract set of postulates. This leads to the concept of categoricalness of postulates.

Categoricalness. A set of postulates is said to be categorical whenever every pair of possible interpretations of the postulates are isomorphic, that is, in 1–1 correspondence throughout their tables of operations. A third simple interpretation of the postulates of Set R will show that these postulates surely are not categorical. Consider the four *combinations* or *sets* of letters formed from the letters α, β, taken two, one, and zero at a time. Let $+$ designate the operation of symmetric difference, performed on these sets, and let $\times$ designate the operation of intersection.* This gives us an example of a Boolean algebra of four elements (the elements of the system in this case being themselves classes or sets). In a similar manner by starting with n letters, examples of a Boolean algebra could be constructed containing as elements the 2^n possible combinations or sets of letters taken $n, n-1, \cdots, 1, 0$ at a time.

In spite of the non-categoricalness of the postulates of Set R, it is easy and instructive to modify Set R so as to obtain a categorical set for a two-element Boolean algebra, or for a four-element Boolean algebra. All we have to do is to add a postulate requiring exactly two elements, or, alternatively, exactly four elements in the system. It is then possible to show (with the aid of our theorems) that the addition and multiplication tables are completely determined. It should be noted, however, that in each case the new postulate makes certain of the original postulates redundant. Thus, in the two-element case, Postulates 5 and 9 become redundant and should be deleted. In the four-element case Postulate 9 is again redundant. In fact, Stone has proved that if the number of elements in a Boolean ring is finite, then a unit element exists; so that Postulate 9 is redundant whenever we require a finite number of elements in the system.

Finally, it might be desirable to outline a proof of the equivalence or interdeducibility of Set R and a standard set of postulates for Boolean algebras, say Huntington's first set† (to be called Set H). This set is expressed in terms of multiplication and Boolean addition (rather than our ring addition) and is well known for the complete duality between the two operations. In order to deduce Set H from Set R, we first can give definitions in the R system of Boolean addition ($\vee$) and of the complement, a', of an element, a (namely, $a \vee b = (a+b)+ab$ and $a' = a+e$). Then we can continue the development of theorems without any difficulty and arrive at all of the postulates of Set H (not already contained as

* The symmetric difference of two sets is the set consisting of all elements belonging to one or the other of two sets, but not both. The intersection of two sets is the set consisting of all elements common to both.

† E. V. Huntington, Sets of independent postulates for the algebra of logic, Transactions of the American Mathematical Society, vol. 5, 1904, pp. 288–309.

postulates in Set R). Conversely, starting with Set H, and referring to theorems proved by Huntington, it is possible to deduce the postulates of Set R, after first defining ring addition in terms of Boolean addition ($a+b=ab'\vee a'b$).

Summary. The postulational approach to Boolean algebra suggested here seems to have certain possibilities for introducing postulational methods and concepts to persons with little previous knowledge of foundations of mathematics or abstract mathematical systems. Thus, it furnishes a simple approach to the study of consistency, independence, and categoricalness of postulates; and it gives practice in easy formal deduction under novel conditions. At the same time, it serves to emphasize the relative nature of the truth of the theorems in any mathematical system. Finally, by examining two equivalent but quite dissimilar sets of postulates, it suggests that there is considerable freedom of choice as to the undefined concepts and fundamental propositions of such a system.

THE TWENTY-SIXTH ANNUAL MEETING OF THE KENTUCKY SECTION

The twenty-sixth annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky on Saturday April 11, 1942, in conjunction with the annual meeting of the Kentucky Academy of Science. Professor L. A. Fair, chairman of the Section, presided.

There were twenty-eight in attendance including the following twenty-one members of the Association: N. B. Allison, P. P. Boyd, M. C. Brown, L. W. Cohen, H. H. Downing, L. A. Fair, Clarence Ford, Tryphena Howard, Fritz John, C. G. Latimer, F. Elizabeth Le Stourgeon, W. L. Moore, Sallie E. Pence, D. W. Pugsley, J. K. Reckzeh, G. G. Roberts, W. F. Smith, D. E. South, Guy Stevenson, Mary E. Williams, H. A. Wright.

A luncheon meeting was held in the Student Union building at which the following officers were elected for next year: Chairman, Charles Hatfield, Georgetown College; Secretary, M. C. Brown, University of Kentucky.

The following papers were presented:

1. "Asymptotic formulae" by Professor Fritz John, University of Kentucky.
2. "On conservative transformations of functions of two variables" by J. D. Rommel, Jr., University of Kentucky, introduced by Professor Cohen.
3. "Mathematical camouflage" by Professor H. H. Downing, University of Kentucky.
4. "Fractional integration" by Mrs. A. S. Howard, University of Kentucky, introduced by Professor John.
5. "The use of the Bernoulli coefficients in proving the binomial theorem" by Professor G. G. Roberts, Berea College.
6. "The net of Möbius" by Dean P. P. Boyd, University of Kentucky.
7. "A mathematical method of testing concave aspherical mirrors" by Pro-

fessor W. L. Moore, University of Louisville.

8. "An almost universal form" by Professor C. G. Latimer, University of Kentucky.

9. "Hermite polynomials" by J. C. Eaves, University of Kentucky, introduced by Professor Cohen.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor John emphasized the importance of asymptotic expansions for numerical calculations. The semi-convergent series for the probability integral, the γ function, and others were discussed as examples.

2. Mr. Rommel discussed double integral transformations of real-valued functions of two variables, $x(\sigma, \tau)$ into real-valued functions of two variables, $y(s, t)$. The formal expressions of the transformation used is given by

$$y(s, t) = \int_0^\infty \int_0^\infty k(s, t, \sigma, \tau)x(\sigma, \tau)d\sigma d\tau; \qquad s, t > 0.$$

Necessary and sufficient conditions that this transformation be conservative were stated by the author.

3. Professor Downing pointed out that criticisms of mathematical difficulties and the severe methods used in presenting the subject matter in mathematics texts, together with the demand for combined courses, unified and correlated courses, survey courses, and popular expositions of mathematics have led to some confusion. Titles of books do not always give an indication of the nature of the content, the same title used by different authors is applied to books carrying different subjects, unlike titles are used with books carrying the same matter. And the chapter headings and section headings are very often imaginative and fantastic.

4. Using the identity

$$\int_0^x \int_0^{\xi_n} \cdots \int_0^{\xi_1} f(\xi_1)d\xi_1 \cdots d\xi_n = \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1}f(\xi)d\xi$$

Mrs. Howard defined $\phi(x)$ to be the n-th integral of $f(x)$ by

$$\phi(x) = \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1}f(\xi)d\xi.$$

She extended the relationship between this and $f(x) = d^n\phi(x)$ by means of the Gamma function and proved some theorems concerning $\phi(x)$.

5. Professor Roberts defined the Bernoulli numbers, proved some special properties and used them in deriving a proof of the binomial theorem.

6. Dean Boyd presented the pioneer work of August Ferdinand Möbius, 1790–1868, mentioning his homogeneous coordinates, directed line segments, cross ratios, spherical trigonometry, twisted curves. His construction of corresponding elements in two collinear or correlative planes by means of the net

was described in detail and theorems on rational and irrational points were demonstrated.

7. In testing a twelve-inch telescope mirror, Professor Moore used the familiar Ronchii test. Trying to get the shape of the shadows of the Ronchiigram, he set up the equations of the shadows as if the source of light were a point. The wire in the grating would cut the cone of rays from the mirror surface and would appear as shadows on the surface. To get the theoretical curves the wire was considered as a line each point of which lay on the normals to the surface. The traces of these normals on the xz plane turned out to be

$$px^2(x-h)^3+pz^2(x-h)^3-d^2x^2(dx-2px+2ph)=0$$

where h is the distance of the line from the optical axis, and d is the distance of the line from the surface of the mirror. When the Ronchii line is on the optical axis, $H=0$ and the trace of the normals on the xz plane consists of a straight line and a circle whose center is at the center of the mirror and whose radius is equal to the projection of the radius of the zone tested on the xz plane.

8. The paper of Professor Latimer was concerned with the integers represented by $f=x^2+y^2+7z^2+7w^2$. By employing known results on the quadratic sub-fields of a quaternion algebra and on the classes of integral sets in such an algebra, he showed that a positive integer is represented by f if and only if it is not in the form $3\cdot 7^n$ or $6\cdot 7^n$.

9. Mr. Eaves showed how a general expression for the Hermite Polynomials may be obtained by using the generating function $\rho-y^2+2xy$ and certain known properties of the polynomials. Other relationships and properties were deduced from these.

D. E. SOUTH, *Secretary*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

CLASSIFICATION OF THE CONICS

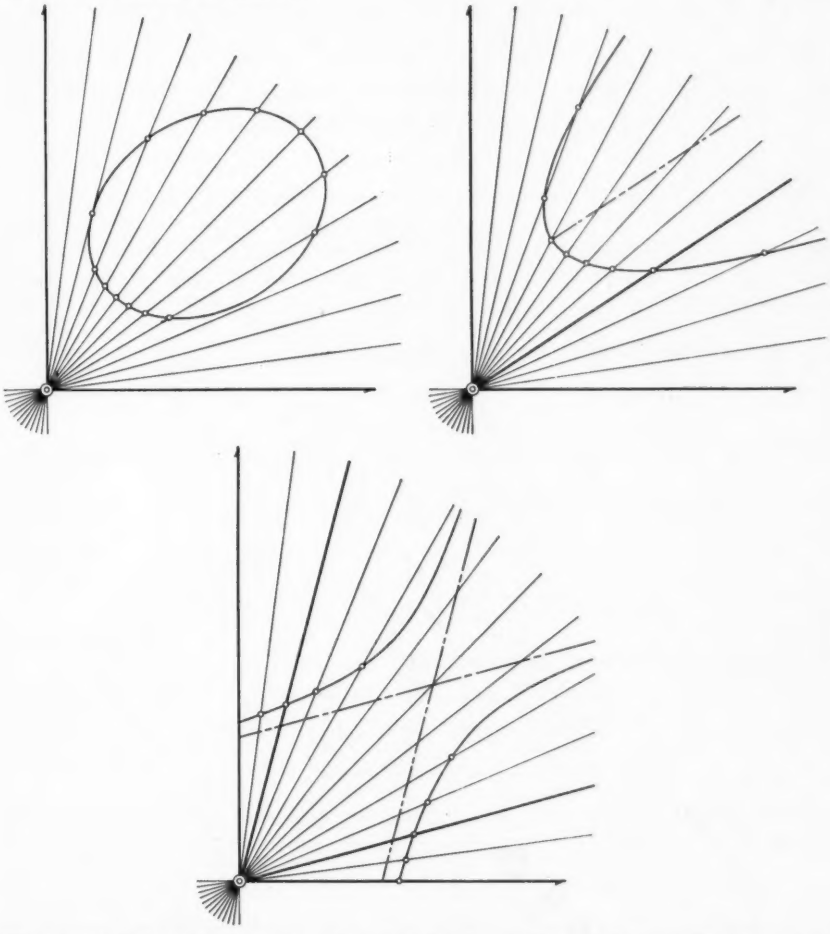
R. C. YATES, U. S. Military Academy

The universal method of obtaining the discriminant relation for the conics by a transformation of axes is tedious, involved, and often omitted in freshman courses. As a first introduction to geometrical invariants, its value is unquestionable. But if the primary object is the identification of the conics, the following approach has several desirable features that warrant its notice.

The three types of conics represented by the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad (D, E \neq 0) \tag{1}$$

are characterized* as follows.



The *parabola* is such that there is *one and only one* line of the pencil $y = mx$ that cuts the curve in only one point (*i.e.*, the line parallel to the axis of the curve).† The *hyperbola* is such that there are *two and only two* distinct lines of the pencil

* If thought desirable, these characterizations may be justified by means of the "standard" forms with which the student is already familiar.

† We consider here "algebraic" points and real lines $y = mx$. A point of tangency, for instance, is counted twice.

$y=mx$ that cut the curve each in only one point (*i.e.*, the lines parallel to the asymptotes of the curve). The *ellipse* is such that there is *no* line of the pencil $y=mx$ that cuts the curve in only one point.

The pencil $y=mx$ meets (1) in points whose abscissas are given by

$$(A + Bm + Cm^2)x^2 + (D + Em)x + F = 0. \tag{2}$$

If some lines of the pencil are to yield but one point of intersection, then

$$A + Bm + Cm^2 = 0,$$

or,*

$$\boxed{m = \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2C}}.$$

For the parabola, only one such line exists. Thus†

$$\boxed{B^2 - 4AC = 0}.$$

For the hyperbola, there are two such lines. Thus‡

$$\boxed{B^2 - 4AC > 0}.$$

For the ellipse, there is no such line. Thus

$$\boxed{B^2 - 4AC < 0}.$$

The characterization fails when $D=E=0$. However, the abscissas of the points of the curve cut out by the pencil are given by

$$(A + Bm + Cm^2)x^2 + F = 0.$$

Since there are always two values of x, real or imaginary, equal and opposite in sign, it is evident that such conics are central.

* If $C=0$, $A\neq 0$, we establish, instead of (2), the corresponding equation for the ordinates of points of intersection of the pencil $x=ny$ and the conic.

† Here, obviously, $m=-B/2C$ is the slope of the axis of the curve. Rotation of axes through the angle arc tan $(-B/2C)$, accordingly, removes the xy term.

‡ The bisector of an angle formed by the two lines here is parallel to an axis of the curve. The angle of rotation to remove the xy term is thus easily obtained.

If $B^2-4AC>0$, the quantity $A+Bm+Cm^2$ changes sign as m ranges through all real values. Thus some lines of $y=mx$ cut the curve and some do not. The conic is thus a *hyperbola*.

If $B^2-4AC<0$, the quantity $A+Bm+Cm^2$ has the same sign for all real values of m. If this sign is opposite that of F, *all* lines $y=mx$ cut the curve. The conic is therefore an *ellipse*. If the sign is the same as that of F, the curve is imaginary.

AN EXISTENCE PROOF FOR LOGARITHMS

F. E. HOHN, University of Arizona

The purpose of this article is to give a simple proof of the following theorem, customarily assumed in college algebra courses:

THEOREM. *Given a real number $b>1$ and any positive number N, then there always exists a real number q, called the logarithm to the base b of N, such that $b^q=N$.*

The method of proof is to *construct the number* q, and to this end we prove first the following preliminary result:

LEMMA. *Given a number $b>1$ and any positive number N, we can construct a sequence of decimal fractions of the form $q_0=p_0$, $q_1=p_0+p_1/10$, $q_2=p_0+p_1/10+p_2/100$, $\cdots$, such that b^{q_n} approaches N as n increases.*

Proof: Since $N>0$ there is a unique integer p_0, not necessarily positive, such that

$$b^{p_0} \leqq N < b^{p_0+1}.$$

Then

$$1 \leqq N/b^{p_0} < b,$$

and hence

$$1 \leqq N^{10}/b^{10p_0} < b^{10}.$$

There is then a unique integer p_1, $0 \leqq p_1 \leqq 9$, such that

$$b^{p_1} \leqq N^{10}/b^{10p_0} < b^{p_1+1}.$$

Then

$$1 \leqq N^{10}/b^{10p_0+p_1} = N^{10}/b^{10q_1} < b,$$

so that

$$1 \leqq N^{100}/b^{100q_1} < b^{10}.$$

There is then a unique integer p_2, $0 \leqq p_2 \leqq 9$, such that

$$b^{p_2} \leqq N^{100}/b^{100q_1} < b^{p_2+1}.$$

Then

$$1 \leqq N^{100}/b^{100q_1+p_2} = N^{100}/b^{100q_2} < b.$$

This process can evidently be continued to obtain

$$1 \leqq N^{10^n}/b^{10^n q_n} < b,$$

which gives

$$1 \leqq N/b^{q_n} < b^{10^{-n}},$$

$$0 \leqq (N/b^{q_n}) - 1 < b^{10^{-n}} - 1$$

$$0 \leqq N - b^{q_n} < b^{q_n}(b^{10^{-n}} - 1) \leqq N(b^{10^{-n}} - 1). \tag{1}$$

The lemma follows at once if we assume that $b^{10^{-n}}$ approaches 1 as the integer n increases. This can be considered a consequence of the fact that $\lim_{x \to 0} b^x = b^0 = 1$, which most students will be willing to grant.

If we now define q and b^q by

$$q = \lim_{n \to \infty} q_n, \qquad b^q = \lim_{n \to \infty} b^{q_n},$$

then $N = b^q$, or $q = \log_b N$.

It is worth noting that as far as most applications of logarithms are concerned we never need to use anything beyond relation (1). For instance, a table accurate to 5 places can be constructed from (1) with $n = 6$.

Although this method of computing logarithms is not of much practical value, it can be carried out without too much labor. With the help of a table of 10th powers, the value of log 3 can be computed to 5 places in about ten minutes.

Note by the Editor. The proof that $\lim_{n \to \infty} b^{10^{-n}} = 1$, though probably not suitable for an elementary class, is quite simple. In the first place, $b^{10^{-n}} > 1$ for all positive n, for if $b^{10^{-n}} \leqq 1$, then $b = (b^{10^{-n}})^{10^n} \leqq 1$ contrary to assumption. Hence if $\lim_{n \to \infty} b^{10^{-n}} \neq 1$, we must have $b^{10^{-n}} > 1 + \epsilon$ for some $\epsilon > 0$ and arbitrarily large n. Then $b = (b^{10^{-n}})^{10^n} > (1+\epsilon)^{10^n} > 1 + 10^n\epsilon$ by the binomial theorem, and this cannot be true for arbitrarily large n. R.J.W.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Differential- und Integralrechnung. By Otto Haupt and Georg Aumann. I Band: *Einführung in die reelle Analysis*, pages 1–196, II Band: *Differentialrechnung*, pages 1–168, III Band: *Integralrechnung*, pages 1–183. Berlin, Walter de Gruyter and Company, 1938. RM. 30.00.

Differential- und Integralrechnung is designed for an analysis course intermediate between the traditional advanced calculus and the modern super-real variables, which begins with boolean algebras, lattices, topology and continues with Banach Spaces and integrals with values therein. Emphasis is placed on intuitive development rather than on ultra-formal argument. Nevertheless, some novel features are included.

In the first half of the first volume, *Introduction to Real Analysis*, the continuum is introduced rapidly (long before the reader gets tired) and the usual limit properties of number sequences and sets are defined and explored. In the second half real functions of one and more real variables are considered. At the end of this half the author has a section on inequalities and convex functions, a subject not generally found in a book of this type.

The second volume, *Differential Calculus*, again, is divided into two parts. In the first part the notion of a derivative is introduced and discussed. After the rules of differentiation are developed, the mean value theorem, Taylor series, what some texts call evaluation of *indeterminates*, the indefinite integral, and circular and hyperbolic functions are considered. These are then illustrated mainly with geometric examples and then the properties of monotone functions are given further attention. In the second half functions of many real variables are considered, with emphasis, at the end, on systems of equations and Jacobians.

The last volume, *Integration*, is the most abstract of the three, which is of course pedagogically justified when one considers the additional sophistication the reader has supposedly acquired on his way to the end. *Measure* and *content*, for the most part in general fields of sets (*Mengenkörper*) are dealt with in the first half. Here we find more emphasis on the notion of *content* than is customary. In the second half there is considered first, integration associated with *content* and second, integration associated with *measure*. The final section on examples and applications begins with properties of the Lebesgue integral and ends with a consideration of bounded k-dimensional *surfaces* in E_n and generalizations to E_n of the theorems of Gauss and Stokes.

SEYMOUR SHERMAN

An Outline of College Algebra. By Gerald E. Moore. N. Y. Barnes and Noble, Inc., 1942. 224 pages. $1.00.

The author states in his preface that this book is intended primarily for students who have had at least one year of high school algebra. After a brief summary (five chapters) of the elementary rules and operations of algebra the author considers those topics generally found in a freshman course in college algebra: solution of equations of various kinds, the binomial theorem, theory of equations, probability, determinants, etc. His method of presenting each of these topics is to state briefly the theory involved and then to give numerous examples illustrating this and methods of solving problems. These varied and well chosen illustrations are one of the best features of the book, the chapters on curve

plotting and quadratics being especially well done. The author is more careful than most about giving definitions, explaining new terminology, and in stating clearly the assumptions used in each problem. He wisely avoids giving proofs of such basic theorems as $(-a)(-b)=ab$. Proofs of these given in texts of this level are generally muddled; here the author frankly states that they are to be accepted by the student as rules which are best proved by the modern methods of abstract algebra.

As a review for someone who has already been over the material this book seems well suited, but the reviewer is dubious of the author's claim that it can be used as a text by students beginning the subject even if they have had a year of elementary algebra. Though complete and well arranged it seems too brief for this, and there are no problems scattered through the book for the student to cut his teeth on. There is a series of eight general exams in the back of the book, but these are not enough to give the student sufficient practice and a real mastery of the subject. An unusual feature is a bibliography of nineteen contemporary college algebra texts giving for each topic studied the pages where it may be found in each of the texts listed.

W. H. DURFEE

Lezioni di Analisi Matematica. Parte Seconda. Quarta Edizione. By Francesco Tricomi. Cedam—Casa Editrici Dott. A. Milani—Padova. 1939—XVIII. 8+355 pages. Lire 60.

This is the second of two volumes. The first volume ended with the indefinite integral. The second begins with the definition of the definite integral in the Riemann sense. Applications are made to ordinary algebraic and transcendental functions. Other topics treated are development in series, Taylor's formula, Simpson's rule, and the introduction of Fourier series.

Differentiation of functions of more than one variable is discussed with application to mean value, maximum and minimum, and Taylor's series. Topics in differential geometry include: tangent, length, osculating plane, curvature, and torsion for twisted curves; singularities and envelopes for plane curves; analytical representation and the Dupin indicatrix for surfaces; correspondence between points in two planes.

Multiple integrals are treated with application to areas and volumes.

In the chapter on ordinary differential equations types discussed are first order, linear and linear with constant coefficients, development in series, and some mention of equations of higher order.

Partial differential equations are discussed with their use in Laplace's equation and harmonic functions, Green's theorem, the problem of the vibrating cord, and the fundamental problem in the calculus of variation.

This volume is written in the same clear style as the first volume and, like it, has no additional exercises included although they are available.

A. H. BLACK

NEW BOOKS RECEIVED

Rapid Review of Elementary Algebra. (Penny Press Series.) By J. T. May. New York, Emerson Books, Inc., 1942. 32 pages. $0.18.

Technical Handbook for Solving Problems in Shop or Factory. By E. H. Lang. New York, Prentice-Hall, Inc., 1942. 4+100 pages. $1.00.

Plane and Spherical Trigonometry. By F. A. Rickey and J. P. Cole. New York, The Dryden Press, 1942. 10+209 pages. $2.25.

Military and Naval Maps and Grids, Their Use and Construction. By. W. W. Flexner and G. L. Walker. New York, Dryden Press, 1942. 96 pages. $1.00.

On Growth and Form. By D'Arcy W. Thompson. Revised Edition. Cambridge, University Press; New York, Macmillan Co., 1942. 1116 pages. $12.50.

Principles of College Algebra. By M. S. Knebelman and T. Y. Thomas. New York, Prentice-Hall, Inc., 1942. 10+380 pages. $2.50.

Differential Equations. By R. P. Agnew. New York, McGraw-Hill Book Co., 1942. 7+341 pages. $3.00.

Metric Methods in Finsler Spaces and in the Foundations of Geometry. By H. Busemann. (Annals of Mathematics Studies, No. 8.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 247 pages. $3.00.

Topics in Topology. By S. Lefschetz. (Annals of Mathematics Studies, No. 10.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 139 pages. $2.00.

Finite Dimentional Vector Spaces. By P. R. Halmos. (Annals of Mathematics Studies, No. 7.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 5+196 pages. $2.35.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 556. *Proposed by C. W. Bruce, Wesleyan College, Macon, Georgia*

The graph of a quadratic function passes through three given points (x_i, y_i). Show that the abscissa of its maximum or minimum is

$$\begin{vmatrix} x_1^2 & y_1 & 1 \\ x_2^2 & y_2 & 1 \\ x_3^2 & y_3 & 1 \end{vmatrix} \div 2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

E 557. *Proposed by V. Thébault, San Sebastián, Spain*

A sphere (S) of constant radius rolls on a fixed sphere (O) in such a way as to pass through a fixed point A. Determine the loci of the centers of similitude of the spheres (S) and (O).

E 558. *Proposed by V. V. Nákladem, Philadelphia*

Let P be any point in the plane of a triangle ABC. Show that the sum of the squares of the areas of the three triangles PBC, PCA, PAB cannot exceed

$$(PA^2 + PB^2 + PC^2)^2/16.$$

E 559. *Proposed by M. A. Sadowsky, Illinois Institute of Technology*

Let a, b, c be non-negative numbers satisfying $a^2+b^2+c^2<1$. Prove that the three planes $x=a$, $y=b$, $z=c$ will cut out from the first octant of the unit sphere $x^2+y^2+z^2=1$ a curvilinear triangle of area

$$\begin{aligned} \tfrac{1}{2}\pi(1-a-b-c) &+ a(\arcsin b_a + \arcsin c_a) \\ &+ b(\arcsin c_b + \arcsin a_b) \\ &+ c(\arcsin a_c + \arcsin b_c) \\ &- \arcsin b_c c_b - \arcsin c_a a_c - \arcsin a_b b_a), \end{aligned}$$

where a_b stands for $a/\sqrt{1-b^2}$, and so on.

E 560. *Proposed by S. H. Gould, University of Toronto*

In the fifth book of his *Laws*, the philosopher Plato, discussing the distribution of land in a colony, seeks a number divisible by every integer from 1 through 10 and chooses 5040. Show in general that if m and n are positive integers with $n<p$, where p is the smallest prime greater than m, then $m!$ is divisible by n except when $m=3$.

SOLUTIONS

A Perfect Cube

E 522 [1942, 335]. *Proposed by V. Thébault, San Sebastián, Spain*

Find the smallest prime radix for which there exists a perfect cube of the form *abcabc*.

Solution by N. G. Gunderson, Cornell University

Let the perfect cube be

$$n^3 = ap^5 + bp^4 + cp^3 + ap^2 + bp + c = (p^3+1)(ap^2+bp+c).$$

If $p^3+1=\prod q_i^{\alpha_i}$, where the q_i are distinct primes, then n must have as a factor $\prod q_i^{\beta_i}$, where β_i is given by

$$\alpha_i = 3(\beta_i - 1) + \gamma_i, \qquad 0 < \gamma_i \leqq 3.$$

Kraïtchik (*Recherches sur la Théorie des Nombres*, vol. II, 1929, pp. 152–156) has tabulated the factors of $(p^3+1)/(p+1)$ for $p<1000$. Also we have $n<p^2$ and $a \neq 0$. These conditions serve to exclude all values of p less than 1000 except 19, 23, 31, 293. For $p=19$ there are four suitable cubes:

1 2 1 1 2 1, 3 14 1 3 14 1, 8 16 8 8 16 8, 17 5 18 17 5 18.

Also solved by D. H. Browne, W. E. Buker, Daniel Finkel, and E. P. Starke.

Orthocentric Tetrahedra

E 523 [1942, 335]. *Proposed by N. A. Court, University of Oklahoma*

With the vertices of a given orthocentric tetrahedron $ABCD$ as centers, spheres are drawn orthogonal to a given sphere (H) concentric with the polar sphere of $ABCD$. Show that the radical planes of (H) with the four spheres considered form a tetrahedron which is orthocentric, and that its orthocenter coincides with that of $ABCD$.

Solution by L. M. Kelly, U. S. Coast Guard Academy

Let us refer to the four spheres as (A), (B), (C), (D). Since (H) is orthogonal to them, the orthocenter H must lie on the radical axis of (A), (B), (C). Also the intersection of the radical planes of (H) and (A), (H), and (B), (H), and (C), is the radical center of (A), (B), (C), (H), and so lies on the radical axis of (A), (B), (C). Hence, if we refer to this point as D', the line HD' is perpendicular to the plane of centers ABC, and so coincides with HD. But this line is also perpendicular to the radical plane of (H) and (D). Defining analogous points A', B', C', we conclude that H is the orthocenter of the tetrahedron $A'B'C'D'$.

Also solved by Howard Eves and the proposer. The proposer remarks that the two tetrahedra are reciprocal with respect to (H).

A Variant of the Eight Queens Problem

E 524 [1942, 336]. *Proposed by R. V. Heath, Wall St., New York*

Write the numbers 9, 10, 11, 12, 13, 14, 15, 16 in one line. Underneath, place the numbers 1, 2, 3, 4, 5, 6, 7, 8 in such an order that the eight sums and eight differences are sixteen different numbers. In how many ways can this be done?

Solution by H. D. Larsen, University of New Mexico

Consider the grid in the XY plane formed by the abscissas 1, 2, $\cdots$, 8 and the ordinates 9, 10, $\cdots$, 16. Each possible pair of values in the desired arrangement are the coordinates of a lattice point occurring in this grid. All points on the same "positive" diagonal represent pairs of numbers having the same sum, and all points on the same "negative" diagonal represent pairs having the same difference. Clearly, the conditions of the problem require the selection of eight lattice points such that no two of them lie on the same vertical line, the same horizontal line, or the same diagonal. This is precisely the classical Eight Queens Problem. (W. W. R. Ball, *Mathematical Recreations and Essays*, eleventh edition, pp. 165–171.) There are 92 different ways of selecting the eight points in the

manner stated. However, we must further exclude those cases in which the sum of one pair of numbers is equal to the difference of another pair, *i.e.*, in which diagonals of opposite kinds through two of the points intersect on the Y axis. Since this property is maintained by reflection, *i.e.*, by reversing the order of the digits, we need only test 46 of the 92 possible orders. Five of the 46 possibilities are found to pass this test; so there are just ten satisfactory arrangements, *viz.*

37286415, 42736815, 27581463, 36824175, 25741863,
51468273, 51863724, 36418572. 57142863, 36814752.

Also solved by Howard and Ruth Eves, and E. T. Frankel. Eight of the ten arrangements were found by the proposer. (See *Judge*, Apr. 1935, p. 23, and May 1935, p. 27.) Jekuthiel Ginsburg (*Scripta Mathematica*, vol. 5 (1938), p. 63) mistook this for the arithmetical form of the Eight Queens Problem itself, as he failed to notice the extra restriction.

A Commensurable Parallelepiped

E 525 [1942, 336]. *Proposed by Maurice Kraïtchik, New School for Social Research, New York City*

Find parallelepipeds with commensurable edges and diagonals.

Solution by E. P. Starke, Rutgers University

If the parallelepiped is to be rectangular, solutions are easily found. Let a, b, c be three mutually perpendicular edges. The diagonals are equal and have length $x=\sqrt{a^2+b^2+c^2}$, whence

$$x^2 - c^2 = a^2 + b^2.$$

For a solution, take a and b to be any two integers not both odd, and put $x = \frac{1}{2}(m+n)$, $c=\frac{1}{2}(m-n)$, where $mn=a^2+b^2$, m and n are of like parity, and $m>n$. For example:

$$a = 1, \quad b = 2, \quad c = 2, \quad x = 3.$$

Any parallelepiped is determined completely when a trihedral angle is known and the lengths of the edges which form that angle. Let the three edges have lengths a, b, c; and let the face angles formed by b and c, c and a, a and b, be respectively α, β, γ. Let d and d' be the diagonals of the face containing angle α and sides b and c. Then

$$d^2 = b^2 + c^2 - 2bc\cos\alpha, \qquad d'^2 = b^2 + c^2 + 2bc\cos\alpha, \tag{1}$$
$$d^2 + d'^2 = 2(b^2 + c^2).$$

The bisecting parallelogram whose sides are a and d has as its diagonals y and z, two diagonals of the parallelepiped. Thus

$$y^2 + z^2 = 2(a^2 + d^2).$$

Corresponding to a and d', we have similarly for the other two diagonals of the

parallelepiped,

$$w^2 + x^2 = 2(a^2 + d'^2)$$

and

$$(2) \qquad w^2 + x^2 + y^2 + z^2 = 4a^2 + 2(d^2 + d'^2) = 4(a^2 + b^2 + c^2).$$

From (1) and analogous results for the other faces, we have

$$(3) \qquad \begin{cases} \cos\alpha = (w^2 + x^2 - y^2 - z^2)/8bc, \\ \cos\beta = (w^2 - x^2 + y^2 - z^2)/8ca, \\ \cos\gamma = (w^2 - x^2 - y^2 + z^2)/8ab. \end{cases}$$

A tentative method for finding parallelepipeds with integral sides and diagonals is suggested by (2) and (3). Take any integers $a \leqq b \leqq c$. Take any of the representations (2) of $4(a^2+b^2+c^2)$ as a sum of four squares with $w \geqq x \geqq y \geqq z \geq 0$. These numbers will correspond to a solution provided each fraction in (3) is less than 1 in absolute value and each angle in (3) is less than the sum of the other two. Examples follow:

(A) $a=b=2$, $c=3$, $w=x=5$, $y=z=3$, $\alpha = \arccos \frac{7}{8}$, $\beta=\gamma=90°$.

(B) $a=b=c=3$, $w=7$, $x=y=5$, $z=3$, $\alpha=\beta=\arccos 5/9$, $\gamma=\arccos 1/9$.

(C) $a=b=4$, $c=1$. Here $4(a^2+b^2+c^2)=132$, which is the sum of four squares in several ways, *viz.*

$$(w, x, y, z) = (11, 3, 1, 1), \quad (9, 5, 5, 1), \quad (9, 7, 1, 1), \quad (8, 6, 4, 4), \quad (7, 7, 5, 3).$$

No parallelepiped results, however, since for each of the first two representations at least one cosine exceeds unity, while for the third all cosines equal unity, and for each of the last two one face angle equals the sum of the other two. (The third is interesting as a limiting case where the vertices all lie on a line; the last two, where the vertices all lie in a plane.) The vertices of the parallelepiped (B), referred to rectangular axes through one vertex, are

$$(0, 0, 0), \quad (3, 0, 0), \quad \left(\frac{10}{3}, \frac{4\sqrt{5}}{3}, 0\right), \quad \left(\frac{1}{3}, \frac{4\sqrt{5}}{3}, 0\right),$$

$$\left(\frac{5}{3}, \frac{2\sqrt{5}}{3}, 2\right), \quad \left(\frac{14}{3}, \frac{2\sqrt{5}}{3}, 2\right), \quad (5, 2\sqrt{5}, 2), \quad (2, 2\sqrt{5}, 2).$$

A Locus Determined by Two Fixed Circles

E 526 [1942, 404]. *Proposed by R. C. Yates, Louisiana State University*

Find the locus of P if the angles formed by the tangents from P to two fixed circles are equal.

Solution by Howard Eves, Syracuse University

It is easily shown, by means of similar triangles, that the distances from P to the two centers are proportional to the respective radii. Hence P describes the

circle of similitude of the given circles.

Also solved by F. A. Alfieri, D. H. Browne, L. M. Kelly, J. Rosenbaum, P. D. Thomas, and J. B. Welchons. Browne remarks that this problem suggests another:

"An angle of given size moves so that its legs are always tangent internally (or externally) to two given circles. Find the locus of the vertex of the angle."

J. H. Butchart and A. K. Waltz interpreted the given problem in this manner, and found a locus consisting of two limaçons, each having double contact with both the given circles. By varying the size of the angle (while keeping the circles fixed), Butchart obtains a family of such limaçons; the locus of nodes is the circle of similitude of the given circles.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4070. *Proposed by P. Erdös, Princeton, N. J.*

Let ρ denote the length of the radius of the inscribed circle of the triangle ABC, let r denote the circumradius and let m denote the length of the longest altitude. Show that $\rho+r \leqq m$.

4071. *Proposed by Harry Langman, Brooklyn, N. Y.*

Setting

$$\frac{1}{1^{t+1}}+\frac{1}{2^{t+1}}+\frac{1}{3^{t+1}}+\cdots+\frac{1}{n^{t+1}}=b_t,$$

show that

$$\begin{vmatrix} b_0, & -b_1, & b_2, \cdots, (-1)^{n-1}b_{n-1} \\ -(n-1), & b_0, & -b_1, \cdots, (-1)^{r-2}b_{n-2} \\ 0, & -(n-2), & b_0, \cdots, (-1)^{n-3}b_{n-3} \\ \cdot & \cdot & \cdot \quad \cdots \quad \cdot \\ 0 & 0 & 0 \qquad b_0 \end{vmatrix} = 1.$$

4072. *Proposed by Richard Bellman, Brooklyn College*

Show that

$$e^x = \frac{(1-x^2)^{1/2}(1-x^3)^{1/3}(1-x^5)^{1/5}\cdots}{(1-x)(1-x^6)^{1/6}(1-x^{10})^{1/10}\cdots}, \qquad |x| < 1,$$

where the exponents in the numerator are integers with an odd number of unrepeated prime factors; and those in the denominator have an even number of unrepeated prime factors.

4073. *Proposed by J. M. Feld, New York*

Let p_1 be the number of partitions of the positive integer N into an odd number of distinct positive integers and let p_2 be the number of partitions of N into an even number of distinct positive integers. For instance, for $N=8$, the odd partitions included in p_1 are 8, 5+2+1, 4+3+1; and the even partitions included in p_2 are 7+1, 6+2, 5+3. Prove that: (1) If $N=n(3n\pm1)/2$, where n is an odd integer, $p_1=p_2+1$. (2) If n is an even integer, $p_1=p_2-1$. (3) If N does not have the above form $p_1=p_2$.

4074. *Proposed by V. Thébault, San Sebastián, Spain*

On the sides of the given triangle ABC directly similar triangles ABC_1, etc., are constructed interiorly. Determine these latter triangles so that $(GA_1)^2+(GB_1)^2+(GC_1)^2$ is a minimum, where G is the centroid of ABC.

SOLUTIONS

Trilinear Polars of Four Points

4018 [1942, 64]. *Proposed by N. A. Court, University of Oklahoma*

If four points taken in the four faces of a tetrahedron are collinear, their trilinear polars for the respective faces cannot be (*a*) coplanar, or (*b*) hyperbolic.

I. *Solution by the Proposer*

Let P, Q, R, S be the given points in the faces BCD, CDA, DAB, ABC of the tetrahedron, and let p, q, r, s be their trilinear polars for the respective triangles.

(a) If p, q, r, s lie in the same plane μ, the harmonic pole M of μ for $DABC$ lies on each of the lines AP, BQ, CR, DS, (N. A. Court, *Modern Pure Solid Geometry*, p. 233, art. 716). Now if the points P, Q, R, S lie on a line h, the vertices of the tetrahedron $DABC$ will be coplanar, for they will all lie in the plane Mh, which is absurd.

(b) A necessary and sufficient condition for the lines p, q, r, s to be hyperbolic is that the lines AP, BQ, CR, DS be hyperbolic, *ibid.*, p. 294, ex. 14 *Mathesis*, 1926, p. 29, Q. 2278. If that is the case, a line d may be drawn through the vertex D meeting the lines AP, BQ, CR.

The line d lies in the plane DAP, hence the trace D' of d in the plane ABC is collinear with A and with the trace X of DP on the edge BC. Similarly the point D' lies on the analogous lines BY and CZ. Now the points X, Y, Z are collinear, for they lie on the line of intersection of the plane Dh with the plane ABC, hence the lines AX, BY, DZ cannot have the point D' in common. Thus the lines AP,

BQ, *CR*, *DS* are not hyperbolic, and therefore neither are the lines p, q, r, s.

Consequence: If four lines passing through the vertices of a tetrahedron are hyperbolic, their four traces in the respectively opposite faces cannot be collinear.

II. *Solution by H. S. M. Coxeter, University of Toronto*

Let the four points lie on the line whose Plücker coordinates, referred to the given tetrahedron, are (l, m, n, L, M, N). The points being distinct, we must have $lmnLMN \neq 0$. In fact, the points are

$$(0, n, -m, L), \qquad (-n, 0, l, M), \qquad (m, -l, 0, N), \qquad (L, M, N, 0),$$

and therefore their trilinear polars for the respective faces are

$$x = 0, \quad \frac{y}{n} - \frac{z}{m} + \frac{t}{L} = 0, \qquad y = 0, \quad \frac{z}{l} - \frac{x}{n} + \frac{t}{M} = 0;$$

$$z = 0, \quad \frac{x}{m} - \frac{y}{l} + \frac{t}{N} = 0; \qquad t = 0, \quad \frac{x}{L} + \frac{y}{M} + \frac{z}{N} = 0.$$

These four lines having Plücker coordinates

$$\left(0, \frac{1}{m}, \frac{1}{n}, \frac{1}{L}, 0, 0\right), \qquad \left(\frac{1}{l}, 0, \frac{1}{n}, 0, \frac{1}{M}, 0\right),$$

$$\left(\frac{1}{l}, \frac{1}{m}, 0, 0, 0, N\right), \qquad \left(0, 0, 0, \frac{1}{L}, \frac{1}{M}, \frac{1}{N}\right),$$

are linearly independent, and so cannot be coplanar or belong to a regulus.

But what *positive* statement can be made regarding them? Their mutual invariants (in the notation of P. W. Wood, *The Twisted Cubic*, Cambridge, England, 1913, p. 21) are

$$\tilde{\omega}_{14} = \tilde{\omega}_{23} = \frac{1}{mM} + \frac{1}{nN} = -\frac{lL}{mMnN},$$

$$\tilde{\omega}_{24} = \tilde{\omega}_{31} = -\frac{mM}{nNlL}, \qquad \tilde{\omega}_{34} = \tilde{\omega}_{12} = -\frac{nN}{lLmM}.$$

These, being proportional to $-(lL)^2 : -(mM)^2 : -(nN)^2$, satisfy the relation

$$(\tilde{\omega}_{23}\tilde{\omega}_{14})^{1/4} + (\tilde{\omega}_{31}\tilde{\omega}_{24})^{1/4} + (\tilde{\omega}_{12}\tilde{\omega}_{34})^{1/4} = 0,$$

which shows that *the four lines are tangents to an infinity of twisted cubics.*

Editorial Note. If one of the four points is at a vertex of $ABCD$, say P is at D, then p may be any straight line in the plane of BCD passing through D. If P is on an edge, say CD, but not at a vertex, then p contains CD and each point on CD has this same polar p with respect to BCD. Suppose now that in (a) no one of the four points is at a vertex, P lies on CD, and Q does not lie on a side of CDA. Since here the four points lie on a straight line h, the point R is on DA, S is on CA, and p, r, s are the straight lines of CD, DA, AC, while q is in the plane

of CDA. Hence in (a) we exclude the case of any one of the four points being on an edge of $ABCD$. In this case the point M of the above argument does not lie on h.

In (b) if one of the four points lies on an edge either at or not at a vertex and the four points lie on a straight line h, an examination of the various cases shows that at least two of p, q, r, s are not skew. In the case mentioned above all four are coplanar. Thus we have to consider only the case where no one of the four points lies on an edge, and we shall show that p, q, r, s cannot be a hyperbolic set. If they could be hyperbolic, then, since P does not lie on a side of BCD, the polar p cannot contain a side or pass through a vertex of BCD, and p meets BC in X' not at B or C. We then determine the point X on BC so that B, X, C, X' is a harmonic set of points, and similarly Y on CA, Z on AB. The straight line d' of X' and Y' meets s as well as p and q, and hence also r by reason of the hypothesis. Thus X', Y', Z' lie on d' and AX, BY, CZ meet in D', the pole of d' with respect to ABC. Also P, Q, R lie on DX, DY, DZ, and hence the plane Dh passes through X, Y, Z. But since D' cannot lie on a side of ABC it is impossible for X, Y, Z to lie on a straight line, and we must reject the hypothesis that p, q, r, s are hyperbolic. This completes the proof of (b) with the above limitation.

We now drop the condition that P, Q, R, S lie on a straight line but retain the above restriction on their location, and suppose that p, q, r, s are hyperbolic. then AP, BQ, CR, DS are hyperbolic. We show first that no two of the latter are coplanar. For, if two, say AP, BQ are coplanar, then their plane cuts DC in a point and the fourth harmonic of this point with respect to C and D lies on both p and q. But this is impossible since p and q are skew. Next we see from the above that the straight line $d = DD'$ meets AP, BQ, CR, DS, and similarly for the straight lines a, b, c defined in the same manner. No two of the latter four lines coincide; for, if $a = d$, then D' is at A which is impossible as indicated above. This completes the proof and shows also that a, b, c, d is a hyperbolic set. We may reverse this argument by supposing that AP, BQ, CR, DS are hyperbolic and show then that p, q, r, s are hyperbolic. Thus there is one and only one straight line d through D meeting AP, BQ, CR; denote by D' its intersection with the plane of ABC; the reader can easily supply the remaining details of the proof. The four straight lines a', b', c', d' are also hyperbolic.

If we allow one of the four points to lie on an edge then it is possible for AP, BQ, CR, DS to be hyperbolic while p, q, r, s is not a hyperbolic set. The condition imposed upon the location of the four points is equivalent to the condition that there shall be a one to one correspondence between each of the four points and its trilinear polar. We may now state the theorem.

If the points P, Q, R, S lie on the planes of the faces BCD, CDA, DAB, ABC of a tetrahedron $ABCD$ so that no one is on an edge, a necessary and sufficient condition that the straight lines of AP, BQ, CR, DS shall be hyperbolic is that the polars p, q, r, s of the corresponding points with respect of the corresponding triangles of the faces be hyperbolic.

Tangents to a Parabola

4019 [1942, 64]. *Proposed by Robin Robinson, Dartmouth College*

Given a triangle ABC. Prove that the bisectors of the interior and exterior angles at C, the side AB and its perpendicular bisector, and the perpendiculars to AC at A and to BC at B, are all tangent to a parabola. Locate its focus.

Solution by the Proposer

The conics confocal at A and B constitute a pencil of line-conics, tangent to the isotropic lines through A and B. The locus of the polar of C with respect to the conics of the pencil is the "11-line conic" of C with respect to the pencil, and is the desired parabola.

The 11-line conic of a point C with respect to the pencil of line-conics tangent to the four sides of a given complete quadrilateral contains the following 11 lines:

(*a*) The 2 fixed lines of the Sturm's Involution formed by pairs of tangents to conics of the pencil drawn from C; these fixed lines are the tangents at C to the two conics of the pencil which pass through C.

(*b*) The 3 sides of the common self-conjugate triangle of the pencil, which are the diagonals of the complete quadrilateral.

(*c*) 6 lines, each of them the harmonic conjugate, with respect to two sides of the complete quadrilateral, of the join of their common vertex to C.

In this case, where the isotropic lines through A and B constitute the complete quadrilateral, its vertices are the common foci A and B, the common imaginary foci lying on the perpendicular bisector of AB, and the circular points at infinity I and J. Hence the 11-line conic contains as tangents:

(*a*) The tangents at C to the two conics of the pencil through C, which are (because of the optical property) *the bisectors of the interior and exterior angles at C.*

(*b*) The common axes of the conics of the pencil (*AB and its perpendicular bisector*), and the line at infinity. Hence the 11-line conic is a *parabola.*

(*c*) The harmonic conjugate of AC with respect to the isotropics through A, *viz., the perpendicular to AC at A*; similarly *the perpendicular to BC at B.* The analogous lines at the imaginary foci we shall not use, but those at I and J are isotropic tangents to the parabola, and hence intersect at its focus F. Since AI and BI separate harmonically CI and FI, and similarly for J, the locus of a point whose joins to A, B and C, F form a harmonic set is a circle through A, B, C, F. If four points on a conic form a harmonic set, the tangents at one pair meet on the join of the other pair. Hence *the focus F may be located by drawing the other tangent to the circle through A, B, C from the point where the tangent at C meets AB.*

Note. It is also worth mentioning that the axis of the parabola is perpendicular to the median drawn from C, and that C and F are reciprocal points ($z' = 1/z$) in an Argand diagram in which A and B are the points $z = \pm 1$.

Editorial Note. An analytic solution is as follows. Let Δ_i, $i = 1, 2, 3, 4$, be four straight lines no two of which are parallel, and intersecting in six distinct points

A_{ij}. Then a parabola $y^2=4ax$ is tangent to the four lines, and we have

$$(1) \qquad \Delta_i\colon\ m_iy - x - am_i^2 = 0; \qquad A_{ij}\colon\ am_im_j,\ a(m_i + m_j).$$

Let Δ_4 be the perpendicular bisector of the segment $A_{13}A_{23}$; then A_{34} lies on the directrix since Δ_3 and Δ_4 are perpendicular tangents, and we find by computation that

$$(2) \qquad m_1 + m_2 = 2m_4 = -2/m_3.$$

Let Δ_1', Δ_2' be respectively the perpendiculars at A_{13}, A_{23} to Δ_1, Δ_2, intersecting in A_{12}'. We then find the coordinates of the latter point to be

$$(3) \qquad A_{12}'\colon\ -a,\ 2a(m_1m_2 - 1)/(m_1 + m_2).$$

Hence the straight line through A_{34} and A_{12}' is the directrix. The slopes k_1 and k_2 of the two bisectors of angle $A_{13}A_{12}'A_{23}$ satisfy the equation

$$(4) \qquad k^2 + 2\left(\frac{m_1m_2 - 1}{m_1 + m_2}\right)k - 1 = 0.$$

The equations of the two tangents to the parabola with the slopes k_1 and k_2 may be written by means of (1) and also their intersection a/k_1k_2, $a(k_1+k_2)/k_1k_2$. But by (4) these coordinates reduce to those in (3). This proves that the two bisectors of angle $A_{13}A_{12}'A_{23}$ are tangents to the parabola.

The two circles having diameters $A_{23}A_{24}$ and $A_{13}A_{14}$ intersect in A_{34} and the focus F. It is also seen geometrically that the straight line of the centers is a tangent, since it is the perpendicular bisector of FA_{34}. Similarly, the perpendicular bisector of FA_{12}' is a tangent. Obviously the midpoints of FA_{34}, FA_{12}' lie on the tangent at the vertex.

Areas of Related Triangles

4016 [1941, 705]. CORRECTED. *Proposed by V. Thébault, San Sebastián, Spain*

The points D, E, F are taken on the sides BC, CA, AB of a triangle ABC, and the points α, β, γ are then taken on the straight lines AD, BE, CF so that $A\alpha/AD=B\beta/BE=C\gamma/CF=k$ and $\alpha D/AD=\beta E/BE=\gamma F/CF=\lambda$. Show that the area σ of triangle $\alpha\beta\gamma$ is given by

$$\sigma = (2S + s)k^2 + (1 - 3k)S = \lambda(2\lambda - 1)S + (1 - \lambda)^2s,$$

where S and s denote the areas of ABC and DEF.

Deduce from this that in a complete quadrilateral the midpoints of the three diagonals are collinear.

Solution by P. W. Allen Raine, Newport News High School

Let the letters above denote the vector coordinates of the respective points.

Then by our hypothesis, we have

$$(A-\alpha)=k(A-D),\qquad (B-\beta)=k(B-E),\qquad (C-\gamma)=k(C-F),$$

from which we get

$$(\alpha-\beta)=(1-k)(A-B)+k(D-E),$$
$$(\beta-\gamma)=(1-k)(B-C)+k(E-F).$$

Therefore

$$\begin{aligned}\sigma &= |\,\alpha-\beta\ \beta-\gamma\,|/2\\ &= (1-k)^2|\,A-B\ B-C\,|/2+k^2|\,D-E\ E-F\,|/2\\ &\quad -k(1-k)\{|\,A-B\ F-E\,|/2+|\,E-D\ B-C\,|/2\}\\ &= (1-k)^2S+k^2s-k(1-k)S\\ &= (2S+s)k^2+(1-3k)S\end{aligned}$$

Setting $k=1-\lambda$ we obtain the second formula.

In applying this to the complete quadrilateral, let the vertices of the quadrilateral be the points A, B, D, E, and let F, an exterior point of AB be the intersection of AB and DE, the point C being the intersection of BD and AE. Then the diagonals of the quadrilateral are AD, BE, and CF with α, β, γ their respective mid-points.

Since $s=0$, for E, F, and D are collinear, we get, by simple substitution of $k=\lambda=\frac{1}{2}$ in the above expressions, $\sigma=0$, *i.e.* α, β, γ are collinear.

Solved also by Howard Eves, using vectors, C. E. Springer, and by the proposer.

A Differential Operator

4017 [1942, 64]. *Proposed by N. H. Anning, University of Michigan*

It is easy to show that $D(D^2+16)$ will annihilate $\cos^4 x+\sin^4 x$ and also $\cos^6 x+\sin^6 x$. Show in general that the same differential operator will annihilate similar expressions with exponents $4k$ and $4k+2$, where k is any positive integer.

Solution by J. A. Bullard, University of Vermont

By well known formulae expressing powers of $\sin x$ and $\cos x$ in terms of cosines of multiple angles (see this MONTHLY, vol. 40, 1933, p. 161) we have,

$$\begin{aligned}\cos^{4k}x+\sin^{4k}x &= \frac{1}{2^{4k}}\left[2\sum_{h=0}^{2k-1}{}_{4k}C_h\cos 2(2k-h)x+{}_{4k}C_{2k}\right.\\ &\qquad \left.+2\sum_{h=0}^{2k-1}(-1)^h{}_{4k}C_h\cos 2(2k-h)x+{}_{4k}C_{2k}\right]\\ &= \frac{1}{2^{4k-1}}\left[2\sum_{h=0}^{k-1}{}_{4k}C_{2h}\cos 4(k-h)x+{}_{4k}C_{2k}\right].\end{aligned}$$

and

$$\cos^{4k+2} x + \sin^{4k+2} x = \frac{1}{2^{4k+2}}\left[2\sum_{h=0}^{2k} {}_{4k+2}C_h \cos 2(2k+1-h)x + {}_{4k+2}C_{2k+1} - 2\sum_{h=0}^{2k}(-1)^h {}_{4k+2}C_h \cos 2(2k+1-h)x + {}_{4k+2}C_{2k+1}\right]$$

$$= \frac{1}{2^{4k+1}}\left[2\sum_{h=0}^{k-1} {}_{4k+2}C_{2h+1} \cos 4(k-h)x + {}_{4k+2}C_{2k+1}\right].$$

From the theory of linear differential equations it is well known that the operator $F(D) = D(D^2+16)(D^2+64)\cdots(D^2+16k^2)$ will annihilate both these expressions containing a constant term and the cosines of the k multiple angles.

Solved also by Edward Fleisher and A. K. Waltz.

A Determinant Evaluation

4023 [1942, 127]. *Proposed by J. A. Greenwood, Duke University*

Find an expression for the determinant of order $2n$

$$\begin{vmatrix} \theta I_n, & A_n \\ A_n, & \theta I_n \end{vmatrix},$$

where θI_n is a square matrix of order n having the variable θ in the principal diagonal and zeros elsewhere, and A_n is a square symmetric matrix of order n with a for each principal diagonal element, unity for the elements in the two parallels immediately above and below this principal diagonal, and zeros elsewhere.

Solution by John Williamson, Queens College

By matrix multiplication we find that

$$\begin{pmatrix} I_n & 0 \\ -\theta^{-1}A_n & I_n \end{pmatrix}\begin{pmatrix} \theta I_n & A_n \\ A_n & \theta I_n \end{pmatrix} = \begin{pmatrix} \theta I_n & A_n \\ 0 & \theta I_n - \theta^{-1}A_n^2 \end{pmatrix}$$

and, by taking determinants of both sides of this matrix equation, that

$$\Delta = \begin{vmatrix} \theta I_n & A_n \\ A_n & \theta I_n \end{vmatrix} = |\theta^2 I_n - A_n^2| = |\theta I_n + A_n|\,|\theta I_n - A_n| \tag{1}$$

$$= (-1)^n |A_n + \theta I_n|\,|A_n - \theta I_n| = (-1)^n f(\theta) f(-\theta),$$

where $(-1)^n f(\theta)$ is the characteristic polynomial of A_n. Therefore Δ can easily be found if its characteristic polynomial $f(\theta)$ can be calculated. In particular, if A_n has the value given in this problem and if

$$P_n(a) = |A_n|, \tag{2}$$

$$|A_n - \theta I_n| = P_n(a-\theta). \tag{3}$$

The determinant $P_n(a)$ is a perpetuant and its value can be calculated from the recursion formula

$$P_n(a) = aP_{n-1}(a) - P_{n-2}(a)$$

with the initial conditions

$$P_2(a) = a^2 - 1, \qquad P_1(a) = a.$$

It is now easy to show by induction that

$$P_n(a) = \sum_{i=0}^{[n/2]} (-1)^i \binom{n-i}{i} a^{n-2i}. \tag{4}$$

Hence, from (1) and (3),

$$\Delta = (-1)^n P_n(a-\theta) P_n(a+\theta), \tag{5}$$

where $P_n(a)$ is defined by (4).

Several generalizations are possible. If each unit in A_n is replaced by b, then Δ has the value

$$(-1)^n b^{2n} P_n\left(\frac{a+\theta}{b}\right) P_n\left(\frac{a-\theta}{b}\right).$$

If A_n is the matrix with a for each element in the leading diagonal, with b for each element immediately above, and with c for each element immediately below the leading diagonal, and all other elements zero,

$$A_n = P_n(a, b, c) = \sum_{i=0}^{[n/2]} (-1)^i \binom{n-i}{i} a^{n-2i} b^i c^i. \tag{6}$$

The corresponding value of Δ is

$$(-1)^n P_n(a-\theta, b, c) P_n(a+\theta, b, c).$$

The matrix whose determinant we have been considering is a particular case of what might be called a block circulant matrix.* We may write

$$R = \begin{pmatrix} \theta I_n & A_n \\ A_n & \theta I_n \end{pmatrix} = E + A_n T, \quad \text{where} \quad T = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

The matrix T is similar to

$$\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$$

and therefore the matrix R is similar to

$$\begin{pmatrix} I_n + A_n & 0 \\ 0 & I_n - A_n \end{pmatrix}.$$

* John Williamson. "The latent roots of a matrix of special type." Bull. Amer. Math. Soc., Aug. 1931. The matrices considered in §3 of this paper are orthogonally equivalent to block circulant matrices, since they can be reduced to that form by a re-arrangement of the rows and the same re-arrangement of the columns.

In passing to the more general case we define T to be an m-rowed square matrix whose elements are n-rowed square matrices. The element in the last row and first column of T is the unit matrix I_n, as are all elements in the diagonal above the leading one. Each other element of T is the zero matrix. If $m=3$,

$$T = \begin{pmatrix} 0 & I_n & 0 \\ 0 & 0 & I_n \\ I_n & 0 & 0 \end{pmatrix}.$$

The matrix T is an mth root of the unit matrix and is similar to the diagonal block matrix

$$[\omega_1 I_n, \omega_2 I_n, \cdots, \omega_m I_n],$$

where $\omega_1, \omega_2, \cdots, \omega_m$ are the m distinct mth roots of unity. If

$$R = R_0 T^0 + R_1 T + \cdots + R_{m-1} T^{m-1},$$

where each R_i is a square matrix of order n, R is similar to the diagonal block matrix

$$\left[\sum_{i=0}^{m-1} \omega_1^i R_i, \sum_{i=0}^{m-1} \omega_2^i R_i, \cdots, \sum_{i=0}^{m-1} \omega_m^i R_i \right]$$

and

$$|R| = \prod_{j=1}^{m} \left| \sum_{i=0}^{m-1} \omega_j^i R_i \right|. \tag{7}$$

For example, if $m=3$,

$$R = \begin{pmatrix} R_0 & R_1 & R_2 \\ R_2 & R_0 & R_1 \\ R_1 & R_2 & R_0 \end{pmatrix}$$

and R is similar to

$$\begin{pmatrix} R_0 + R_1 + R_2 & 0 & 0 \\ 0 & R_0 + \omega R_1 + \omega^2 R_2 & 0 \\ 0 & 0 & R_0 + \omega^2 R_1 + \omega R_2 \end{pmatrix}$$

while

$$|R| = |R_0 + R_1 + R_2|\,|R_0 + \omega R_1 + \omega^2 R_2|\,|R_0 + \omega^2 R_1 + \omega R_2|.$$

If $R_2 = \theta I_n$ and R_i is obtained from A_n by replacing a by a_i,

$$\left| \sum_{i=0}^{m-1} \omega_j^i R_i \right| = \left| \theta I_n + \sum_{i=1}^{m-1} \omega_j^i R_i \right| = P_n\left(\theta + \sum_{i=1}^{m-1} \omega_j^i a_i, \sum_{i=1}^{m-1} \omega_j^i, \sum_{i=1}^{m-1} \omega_j^i \right)$$

$$P_n\left(\theta + \sum_{i=1}^{m-1} a_i, m-1, m-1\right) = (m-1)^n P_n\left(\frac{\theta + \sum a_j}{m-1}\right), \quad \omega_j = 1,$$

$$P_n\left(\theta + \sum_{i=1}^{m-1} \omega_j^i a_i, -1, -1\right) = (-1)^n P_n\left(-\theta - \sum_{i=1}^{m-1} \omega_j^i a_i\right), \quad \omega_j \neq 1.$$

Hence from (7) we have

$$(8)\quad |R| = (m-1)^n(-1)^{n(m-1)} P_n\left(\frac{\theta + \sum_{i=1}^{m-1} a_i}{m-1}\right) \prod_{j=2}^{m-1} P_n\left(-\theta - \sum_{i=1}^{m-1} \omega_j^i a_i\right),$$

where $\omega_2, \omega_3, \cdots, \omega_m$ are the $m-1$ mth roots of unity distinct from unity itself. If $a_i = a$, $i = 1, 2, \cdots, m-1$, (8) becomes

$$|R| = (m-1)^n(-1)^{n(m-1)} P_n\left(a + \frac{\theta}{m-1}\right)[P_n(a-\theta)]^{m-1}.$$

In particular if $m = 3$,

$$\begin{vmatrix} I_n & A_n & A_n \\ A_n & I_n & A_n \\ A_n & A_n & I_n \end{vmatrix} = 2^n P_n(a + \tfrac{1}{2}\theta)[P_n(a-\theta)]^2.$$

C. R. Cassity gave a formula expressing the given determinant in terms of similar determinants for smaller values of n.

Similar Triangles on a Polygon

4025 [1942, 128]. *Proposed by V. Thébault, San Sebastián, Spain*

Let $A_1', A_2', \cdots, A_{2n}'$ be the vertices of equilateral triangles constructed externally (or internally) on the sides $A_1A_2, A_2A_3, \cdots, A_{2n}A_1$ of a plane polygon of $2n$ sides $(P) \equiv A_1A_2 \cdots A_{2n}$, and $M_1, M_2, \cdots, M_n$ be the midpoints of the principal diagonals $A_1A_{n+1}, A_2A_{n+2}, \cdots, A_nA_{2n}$ of (P). The midpoints $M_1', M_2', \cdots, M_n'$ of the principal diagonals $A_1'A_{n+1}', A_2'A_{n+2}', \cdots, A_n'A_{2n}'$ of the polygon $(P') \equiv A_1'A_2', \cdots, A_{2n}'$ are the vertices of equilateral triangles constructed upon the sides of the polygon $(p) \equiv M_1M_2 \cdots M_n$.

Generalize by replacing the equilateral triangles by similar isosceles triangles.

I. *Solution by J. R. Musselman, Western Reserve University*

The problem may be generalized by replacing everywhere equilateral triangles by triangles similar to a given triangle $X_1X_2X_3$. If we denote the vertices of the polygon P by the complex coordinates $a_i (i = 1, 2 \cdots 2n)$; then the conditions that the triangles $A_i' A_{i+1} A_i$ and $A_{n+i}' A_{n+i+1} A_{n+i}$, constructed externally upon the sides $A_{i+1} A_i$ and $A_{n+i+1} A_{n+i}$ respectively, be similar to the triangle $X_1X_2X_3$ are

$$\begin{vmatrix} a_i' & a_{i+1} & a_i \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = 0; \quad \text{and} \quad \begin{vmatrix} a_{n+i}' & a_{n+i+1} & a_{n+i} \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

[all subscripts taken mod $(2n)$].

Adding these two determinants and dividing by 2, we have

$$\begin{vmatrix} \frac{1}{2}(a_i' + a_{n+i}') & \frac{1}{2}(a_{i+1} + a_{n+i+1}) & \frac{1}{2}(a_i + a_{n+i}) \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Now $\frac{1}{2}(a_i' + a_{n+i}')$ is the coordinate of the midpoint, M_i', of $A_i'\ A_{n+i}'$, $\frac{1}{2}(a_{i+1} + a_{n+i+1})$ and $\frac{1}{2}(a_i + a_{n+i})$ are the coordinates of the midpoints of $A_{i+1}A_{n+i+1}$ and A_iA_{n+i} respectively. Hence we have proved that the triangle $M_i'\ M_{i+1}M_i$ is similar to the triangle $X_1X_2X_3$. It is obvious that the problem is true if we replace midpoints everywhere, by points dividing their respective segments in the ratio $\lambda:\mu$. A similar argument would show the truth of the problem if the similar triangles all be constructed internally upon the sides of the polygon P.

In the special case of equilateral triangles when $n=2$ the figure $M_1'\ M_2M_2'\ M_1$ is a rhombus with angles $2\pi/3$ and $\pi/3$; when $n=3$, since M_i' are vertices of equilateral triangles constructed externally on $M_1M_2M_3$, the lines $M_1'M_3M_2'M_1$ and $M_3'M_2$ are all equal in length and intersect at angles of $2\pi/3$ at a point known as the Fermat point of the triangle $M_1M_2M_3$. The other Fermat point arises when the equilateral triangles are constructed internally on the sides of $M_1M_2M_3$. If the points corresponding to M_i' be called M'' then the hexagons formed from the six points $M_i'M''$ have been discussed by the writer. See *American Journal of Mathematics*, vol. 57 (1935), p. 504.

II. *Solution by Howard Eves, Bryan, Texas*

Let us generalize at the outset by replacing the set of equilateral triangles constructed upon the sides of (P) by a set of directly similar triangles so constructed, letting A_1', A_2', $\cdots$, A_{2n}' be vertices of these triangles. Now, by virtue of the fundamental theorem: "If lines joining corresponding points of two directly similar curves be divided proportionally, the locus of the dividing points is a curve directly similar to the given curves," it follows that the set of triangles $M_1'M_1M_2$, etc., constructed upon the sides of (p) are all directly similar to those on the sides of (P).

Solved also by the proposer using rectangular coordinates.

Editorial Note. The proof by Musselman is essentially a proof of the above fundamental theorem, and it may be put in the following form to show this. Let C and C' be two directly similar curves in the same plane; A, A' two fixed points

on the respective curves which correspond in this similitude; and P, P' two corresponding variable points on the curves. Let the origin O of complex coordinates divide $A'A$ in the ratio $k:1$; let M be the point dividing the straight line segment $P'P$ in the same ratio; and let the complex coordinates of A, A', P, P' be a, $-ka$, $a+p$, $-ka+p'$. Then $p'=\lambda p$, where λ is a complex constant for this similitude. The coordinate of M is

$$m = [(-ka + \lambda p) + k(a + p)]/(1 + k) = [(\lambda + k)/(1 + k)]p.$$

Hence the curve described by M is similar to C.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

The New Jersey State Teachers College, Upper Montclair, has been designated by the United States Office of Education as one of three war information centers in its state. Various visual and teaching aids incident to a war-time program, chiefly in the secondary school field, may be obtained by writing to the Teaching Aids Service at this institution.

Dean L. P. Eisenhart of Princeton University has been appointed executive officer of the American Philosophical Society to succeed Professor Emeritus E. G. Conklin of Princeton University who is now president of the Society.

J. B. Adkins of Phillips Exeter Academy has been appointed an instructor in navigation in the Naval Reserve Midshipmen's School, New York, and Dr. D. M. Krabill of Potomac State School, West Virginia, is also an instructor in the Midshipmen's School.

Assistant Professor H. W. Bailey of the University of Illinois has been promoted to an associate professorship.

Professor B. L. Beegle of Seattle Pacific College has been appointed dean of the college.

Dr. E. E. Blanche of Michigan State College has been appointed statistical director at the Curtiss-Wright Corporation, Buffalo, N. Y.

Dr. H. E. Burns of Indiana University Extension, East Chicago, Indiana, has been promoted to an assistant professorship.

Professor J. H. Butchart of William Woods College, Missouri, has been appointed an assistant professor at Grinnell College.

Associate Professor H. V. Craig of the University of Texas has been promoted to a professorship.

R. W. Erickson of Hibbing Junior College is on leave and is a second lieutenant in the Army Specialist Corps.

Professor H. B. Evans of the University of Pennsylvania has retired with the title professor emeritus.

Associate Professor H. P. Evans of the University of Wisconsin has been promoted to a professorship.

Dr. F. A. Ficken of Cornell University has been appointed to a professorship at the University of Tennessee.

Assistant Professor Frances Harshbarger of Kent State University has been promoted to an associate professorship.

Associate Professor Sara L. Nelson of Georgia State College for Women has been promoted to a professorship and is head of the department of mathematics.

The Reverend R. E. O'Connor has been appointed to a professorship at Loyola College, Montreal, Quebec.

Dr. E. W. Paxson of Wayne University has been promoted to an assistant professorship.

Dr. W. T. Puckett of the University of California at Los Angeles has been promoted to an assistant professorship.

Assistant Professor Helen G. Russell of Wellesley College is on leave for 1942–43 and is at the University of California.

Professor W. J. Rusk of Grinnell College has retired.

Professor C. B. Upton of Teachers College, Columbia University, has retired with the title professor emeritus.

Professor D. V. Widder of Harvard University has been appointed chairman of the department, succeeding Professor M. H. Stone.

The following appointments to instructorships are announced:

University of California: Dr. Dorothy L. Bernstein
Princeton University: Paul Brock, part-time
Purdue University: N. J. Fine
Randolph Field, Texas: Dr. C. L. Seebeck
State Teachers College, Mankato, Minn.: C. J. Kirchen
Williams College: E. F. Gillette
University of Wisconsin: P. B. Norman

A recent message received through the American Red Cross announces the death in Germany on July 5, 1942, of Professor Oskar Bolza at the age of eighty-

five. He was a Reader in Mathematics at Johns Hopkins University in 1888–89, Associate at Clark University 1889–93, Associate Professor at the University of Chicago 1893–94 and Professor 1894–1910. For many years past he has been non-resident professor living in Freiburg.

As this issue was going to press word came of the death of Professor E. R. Hedrick on February 3, 1943.

THE WAR POLICY COMMITTEE

The American Mathematical Society and the Mathematical Association of America have appointed a joint War Policy Committee which will act for mathematicians in the many complex problems which will arise during the war period. Presidents W. D. Cairns and M. H. Stone, of the Association and Society respectively, will be ex officio members of the Committee and the other members are: Professors G. C. Evans, L. M. Graves, Marston Morse, Dr. Warren Weaver, and Professor G. T. Whyburn. Professor Stone will act as Chairman, at least during the initial stages of the work of the Committee.

The chief problems which will immediately concern this Committee are the deferment of mathematicians and the formulation of further plans to alleviate the serious shortage of teachers of mathematics which will confront many institutions upon the inauguration of the Army and Navy Training Program.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

The following is a list of the Sections of the Associations, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Notre Dame, Ind., April 9–10, 1943
INDIANA, Notre Dame, April 9–10, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brookiyn, N. Y., May 8, 1943
MICHIGAN, Notre Dame, Ind., April 9–10, 1943
MINNESOTA
MISSOURI, Kansas City
NEBRASKA
NORTHERN CALIFORNIA
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May 7, 1943

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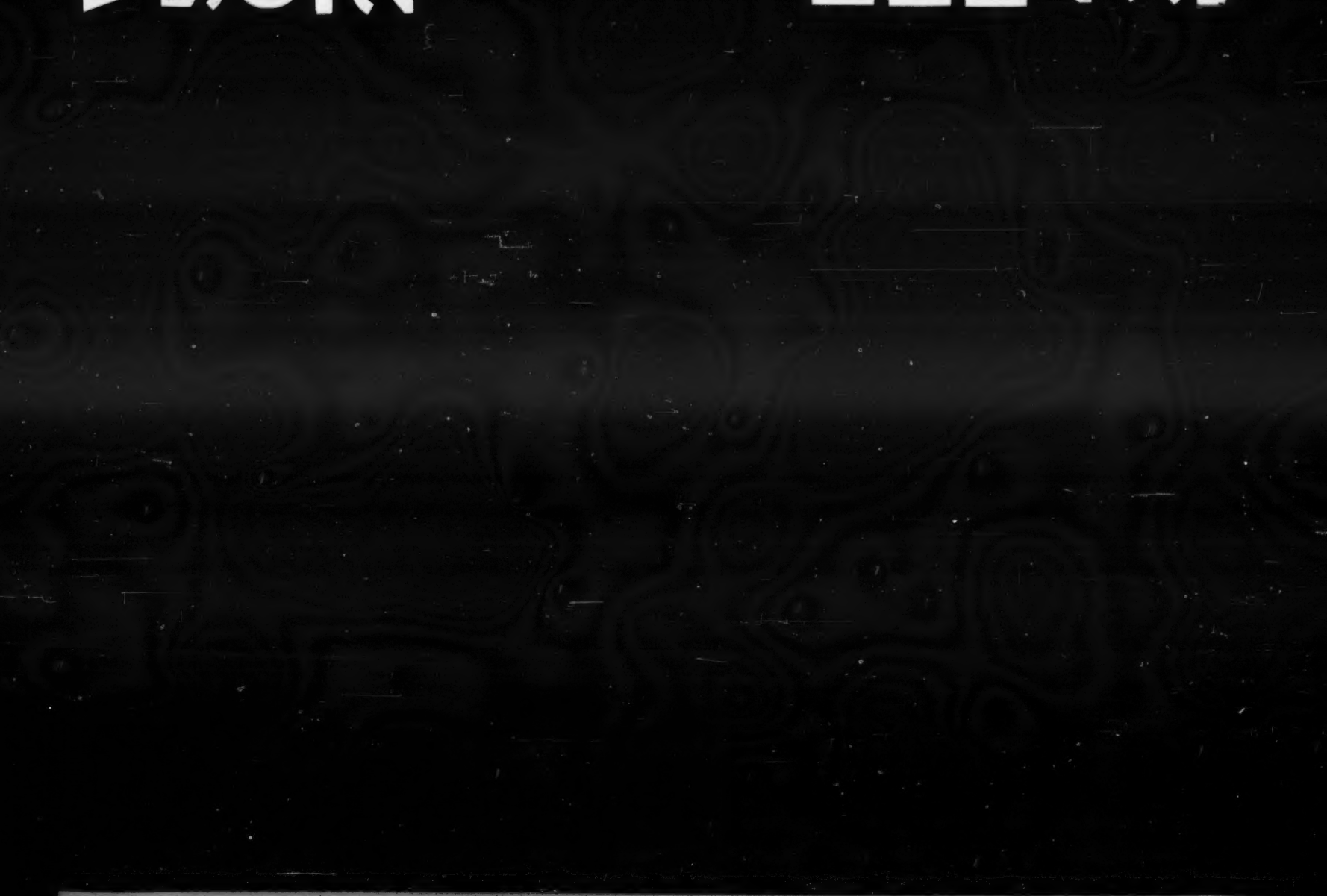

MATHEMATICAL ASSOCIATION OF AMERICA

The office of the Secretary-Treasurer of the Mathematical Association, which has been at Oberlin, Ohio, since the organization in 1915, has been moved to Ithaca, New York, as of January 1, 1943, with the beginning of Professor Walter B. Carver's term as Secretary-Treasurer. All business correspondence, all subscriptions to the

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